(spin 2) = (spin 1)^2

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1.0 Basic idea

- Strong nuclear, Weak nuclear and Electromagnetic forces described by Yang-Mills gauge theory (non-abelian generalisation of Maxwell).
  - Gluons, W, Z and photons have spin 1.

- Gravitational force described by Einstein’s general relativity.
  - Gravitons have spin 2.

- But maybe $GRAVITY = (YANG - MILLS)^2$

- If so, gravitational symmetries should follow from those of Yang-Mills
1.1 Gravity as square of Yang-Mills

- A recurring theme in attempts to understand the quantum theory of gravity and appears in several different forms:
- Closed states from products of open states and KLT relations in string theory [Kawai:1985, Siegel:1988],
- On-shell $D = 10$ Type IIA and IIB supergravity representations from on-shell $D = 10$ super Yang-Mills representations [Green:1987],
1.2 Local and global symmetries from Yang-Mills

- LOCAL SYMMETRIES: general covariance, local lorentz invariance, local supersymmetry, local p-form gauge invariance

  [arXiv:1408.4434
  A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]

- GLOBAL SYMMETRIES eg $G = E_7$ in $D = 4, \mathcal{N} = 8$ supergravity

  A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]

- YANG-MILLS SPIN-OFF (interesting in its own right): Unified description of $(D = 3; \mathcal{N} = 1, 2, 4, 8), (D = 4; \mathcal{N} = 1, 2, 4), (D = 6; \mathcal{N} = 1, 2), (D = 10; \mathcal{N} = 1)$ Yang-Mills in terms of a pair of division algebras $(\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}}), n = D - 2$

  [arXiv:1309.0546
  A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]

- GENERALIZED SELF-MIRROR CONDITION AND VANISHING TRACE ANOMALIES
1.3 Product?

Although much of the squaring literature invokes taking a product of left and right Yang-Mills fields

\[ A_\mu(x)(L) \otimes A_\nu(x)(R) \]

it is hard to find a conventional field theory definition of the product. Where do the gauge indices go? Does it obey the Leibnitz rule

\[ \partial_\mu(f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g) \]

If not, why not?
Here we present a $G_L \times G_R$ product rule:

$$[A_{\mu}^i (L) \star \Phi_{ii'} \star A_{\nu}^{i'} (R)](x)$$

where $\Phi_{ii'}$ is the “spectator” bi-adjoint scalar field introduced by Hodges [Hodges:2011] and Cachazo et al [Cachazo:2013] and where $\star$ denotes a convolution

$$[f \star g](x) = \int d^4y f(y)g(x - y).$$

Note $f \star g = g \star f, (f \star g) \star h = f \star (g \star h)$, and, importantly obeys

$$\partial_\mu (f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g)$$

and not Leibnitz

$$\partial_\mu (f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$
1.5 Gravity/Yang-Mills dictionary

For concreteness we focus on

- $\mathcal{N} = 1$ supergravity in $D = 4$, obtained by tensoring the $(4 + 4)$ off-shell $\mathcal{N}_L = 1$ Yang-Mills multiplet $(A_\mu(L), \chi(L), D(L))$ with the $(3 + 0)$ off-shell $\mathcal{N}_R = 0$ multiplet $A_\mu(R)$.

- Interestingly enough, this yields the new-minimal formulation of $\mathcal{N} = 1$ supergravity [Sohnius:1981] with its $12+12$ multiplet $(h_{\mu\nu}, \psi_\mu, V_\mu, B_{\mu\nu})$.

- The dictionary is,

\[
Z_{\mu\nu} \equiv h_{\mu\nu} + B_{\mu\nu} = A_\mu^i(L) \star \Phi_{ii'} \star A_\nu^{i'}(R)
\]

\[
\psi_\nu = \chi^i(L) \star \Phi_{ii'} \star A_\nu^{i'}(R)
\]

\[
V_\nu = D^i(L) \star \Phi_{ii'} \star A_\nu^{i'}(R),
\]
1.6 Yang-Mills symmetries

- The left supermultiplet is described by a vector superfield $V^i(L)$ transforming as

$$\delta V^i(L) = \Lambda^i(L) + \bar{\Lambda}^i(L) + f^i_{jk} V^j(L) \theta^k(L) + \delta_{(a,\lambda,\epsilon)} V^i(L).$$

Similarly the right Yang-Mills field $A^i\nu(R)$ transforms as

$$\delta A^i\nu(R) = \partial_\nu \sigma^i(R) + f^{i' j' k'} A^j(R) \theta^{k'}(R) + \delta_{(a,\lambda)} A^i(R).$$

and the spectator as

$$\delta \Phi_{ij'} = - f^{j' ik} \Phi_{ji} \theta^k(L) - f^{j' i' k'} \Phi_{ij'} \theta^{k'}(R) + \delta_a \Phi_{ii'}.$$

Plugging these into the dictionary gives the gravity transformation rules.
1.7 Gravitational symmetries

\[ \delta Z_{\mu \nu} = \partial_\nu \alpha_\mu (L) + \partial_\mu \alpha_\nu (R), \]
\[ \delta \psi_\mu = \partial_\mu \eta, \]
\[ \delta V_\mu = \partial_\mu \Lambda, \]

where

\[ \alpha_\mu (L) = A_\mu^i (L) \star \Phi_{ii^\prime} \star \sigma^{i^\prime} (R), \]
\[ \alpha_\nu (R) = \sigma^i (L) \star \Phi_{ii^\prime} \star A_{\nu}^{i^\prime} (R), \]
\[ \eta = \chi^i (L) \star \Phi_{ii^\prime} \star \sigma^{i^\prime} (R), \]
\[ \Lambda = D^i (L) \star \Phi_{ii^\prime} \star \sigma^{i^\prime} (R), \]

illustrating how the local gravitational symmetries of general covariance, 2-form gauge invariance, local supersymmetry and local chiral symmetry follow from those of Yang-Mills.
We also recover from the dictionary the component supersymmetry variation of [Sohnius:1981],

\[ \delta_{\epsilon} Z_{\mu\nu} = -4i\bar{\epsilon}\gamma_\nu \psi_\mu, \]
\[ \delta_{\epsilon} \psi_\mu = -\frac{i}{4} \sigma^{k\lambda} \epsilon \partial_k g_{\lambda\mu} + \gamma_5 \epsilon V_\mu \]
\[ \quad - \gamma_5 \epsilon H_\mu - \frac{i}{2} \sigma_{\mu\nu} \gamma_5 \epsilon H^\nu, \]
\[ \delta_{\epsilon} V_\mu = -\epsilon \gamma_{\mu} \sigma^{\kappa\lambda} \gamma_5 \partial_\kappa \psi_\lambda. \]
New minimal supergravity also admits an off-shell Lorentz multiplet \((\Omega_{\mu ab}^{-}, \psi_{ab}, -2V_{ab}^{\mp})\) transforming as

\[
\delta V^{ab} = \Lambda^{ab} + \bar{\Lambda}^{ab} + \delta_{(a, \lambda, \epsilon)} V^{ab}. \tag{1}
\]

This may also be derived by tensoring the left Yang-Mills superfield \(V^i(L)\) with the right Yang-Mills field strength \(F^{abi'}(R)\) using the dictionary

\[
V^{ab} = V^i(L) \star \Phi_{ii'} \star F^{abi'}(R),
\]

\[
\Lambda^{ab} = \Lambda^i(L) \star \Phi_{ii'} \star F^{abi'}(R).
\]
1.10 Bianchi identities

- The corresponding Riemann and Torsion tensors are given by
  \[ R^+_{\mu\nu\rho\sigma} = -F^{\nu\rho\sigma}_{\mu\nu}\Phi_{\mu\nu} \Phi_{\mu\nu} \Phi_{\rho\sigma} R = R^-_{\rho\sigma\mu\nu}. \]
  \[ T^+_{\mu\nu\rho} = -F_{[\mu\nu}\Phi_{\mu\nu}\Phi_{\mu\nu} A_{\rho]} R = -A_{\rho} R = -T^-_{\mu\nu\rho}. \]
- One can show that (to linearised order) the gravitational Bianchi identities follow from those of Yang-Mills
  \[ D_{[\mu}(L) F_{\nu\rho]} I(L) = 0 = D_{[\mu}(R) F_{\nu\rho]} I'(R) \]
1.11 To do

- Convoluting the off-shell Yang-Mills multiplets \((4 + 4, \mathcal{N}_L = 1)\) and \((3 + 0, \mathcal{N}_R = 0)\) yields the \(12 + 12\) new-minimal off-shell \(\mathcal{N} = 1\) supergravity.

- We expect that convoluting the off-shell general multiplet \((8 + 8, \mathcal{N}_L = 1)\) and \((3 + 0, \mathcal{N}_R = 0)\) yields the \(24 + 24\) non-minimal off-shell \(\mathcal{N} = 1\) supergravity [Breitenlohner:1977].

- We expect convoluting \((4 + 4, \mathcal{N}_L = 1)\) and \((4 + 4, \mathcal{N}_R = 1)\) yields the \(32 + 32\) minimal off-shell \(\mathcal{N} = 2\) supergravity [Fradkin:1979, deWit:1979, Breitenlohner:1979, Breitenlohner:1980]. The latter would involve bosons from the product of left and right fermions.

- Clearly two important improvements would be to generalise our results to the full non-linear transformation rules and to address the issue of dynamics as well as symmetries.
2.1 Division algebras

- Mathematicians deal with four kinds of numbers, called Division Algebras.
- The Octonions occupy a privileged position:

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Imaginary parts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reals</td>
<td>ℝ</td>
<td>0</td>
</tr>
<tr>
<td>Complexes</td>
<td>ℂ</td>
<td>1</td>
</tr>
<tr>
<td>Quaternions</td>
<td>ℍ</td>
<td>3</td>
</tr>
<tr>
<td>Octonions</td>
<td>ℬ</td>
<td>7</td>
</tr>
</tbody>
</table>

Table: Division Algebras
2.2 Lie algebras

They provide an intuitive basis for the exceptional Lie algebras:

<table>
<thead>
<tr>
<th>Classical algebras</th>
<th>Rank</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$n$</td>
<td>$(n + 1)^2 - 1$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$n$</td>
<td>$n(2n + 1)$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$n$</td>
<td>$n(2n + 1)$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$n$</td>
<td>$n(2n - 1)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Exceptional algebras</th>
<th>Rank</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_6$</td>
<td>6</td>
<td>78</td>
</tr>
<tr>
<td>$E_7$</td>
<td>7</td>
<td>133</td>
</tr>
<tr>
<td>$E_8$</td>
<td>8</td>
<td>248</td>
</tr>
<tr>
<td>$F_4$</td>
<td>4</td>
<td>52</td>
</tr>
<tr>
<td>$G_2$</td>
<td>2</td>
<td>14</td>
</tr>
</tbody>
</table>

Table: Classical and exceptional Lie algebras
2.3 Magic square

- Freudenthal-Rozenfeld-Tits magic square

<table>
<thead>
<tr>
<th>$A_L/A_R$</th>
<th>R</th>
<th>C</th>
<th>H</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>$A_1$</td>
<td>$A_2$</td>
<td>$C_3$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>C</td>
<td>$A_2$</td>
<td>$A_2 + A_2$</td>
<td>$A_5$</td>
<td>$E_6$</td>
</tr>
<tr>
<td>H</td>
<td>$C_3$</td>
<td>$A_5$</td>
<td>$D_6$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>O</td>
<td>$F_4$</td>
<td>$E_6$</td>
<td>$E_7$</td>
<td>$E_8$</td>
</tr>
</tbody>
</table>

**Table**: Magic square

- Despite much effort, however, it is fair to say that the ultimate physical significance of octonions and the magic square remains an enigma.
Octonion $x$ given by
$$x = x^0 e_0 + x^10 e_1 + x^2 e_2 + x^3 e_3 + x^4 e_4 + x^5 e_5 + x^6 e_6 + x^7 e_7,$$
One real $e_0 = 1$ and seven $e_i, i = 1, \ldots, 7$ imaginary elements, where $e^*_0 = e_0$ and $e^*_i = -e_i$.

The imaginary octonionic multiplication rules are,
$$e_i e_j = -\delta_{ij} + C_{ijk} e_k \quad [e_i, e_j, e_k] = 2Q_{ijkl} e_l$$

$C_{mnp}$ are the octonionic structure constants, the set of oriented lines of the Fano plane.

$$\{ijk\} = \{124, 235, 346, 457, 561, 672, 713\}.$$

$Q_{ijkl}$ are the associators the set of oriented quadrangles in the Fano plane:

$$ijkl = \{3567, 4671, 5712, 6123, 7234, 1345, 2456\},$$

$$Q_{ijkl} = -\frac{1}{3!} C_{mnp} \epsilon_{mnpijkl}.$$
The Fano plane has seven points and seven lines (the circle counts as a line) with three points on every line and three lines through every point.
Gino Fano (5 January 1871 to 8 November 1952) was an Italian mathematician. He was born in Mantua and died in Verona. Fano worked on projective and algebraic geometry; the Fano plane, Fano fibration, Fano surface, and Fano varieties are named for him. Ugo Fano and Robert Fano were his sons.
2.8 Cayley-Dickson

- **Octonion**
  \[ O = O^0 e_0 + O^1 e_1 + O^2 e_2 + O^3 e_3 + O^4 e_4 + O^5 e_5 + O^6 e_6 + O^7 e_7 = H(0) + e_3 H(1) \]

- **Quaternion**
  \[ H(0) = O^0 e_0 + O^1 e_1 + O^2 e_2 + O^4 e_4 \quad H(1) = O^3 e_0 - O^7 e_1 - O^5 e_2 + O^6 e_4 \]
  \[ H(0) = C(00) + e_2 C(10) \quad H(1) = C(01) + e_2 C(11) \]

- **Complex**
  \[ C(00) = O^0 e_0 + O^1 e_1 \quad C(01) = O^3 e_0 - O^7 e_1 \]
  \[ C(10) = O^2 e_0 - O^4 e_1 \quad C(11) = -O^5 e_0 - O^6 e_1 \]
  \[ C(00) = R(000) + e_1 R(100) \quad C(01) = R(001) + e_1 R(101) \]
  \[ C(10) = R(010) + e_1 (110) \quad C(11) = R(011) + e_1 R(111) \]

- **Real**
  \[ R(000) = O^0 \quad R(100) = O^1 \quad R(001) = O^3 \quad R(101) = -O^7 \]
  \[ R(010) = O^2 \quad R(110) = -O^4 \quad R(011) = -O^5 \quad R(111) = -O^6 \]
2.7 Division algebras

- Division: $ax+b=0$ has a unique solution
- Associative: $a(bc)=(ab)c$
- Commutative: $ab=ba$

<table>
<thead>
<tr>
<th>A</th>
<th>construction</th>
<th>division?</th>
<th>associative?</th>
<th>commutative?</th>
<th>ordered?</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>$R$</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>C</td>
<td>$R + e_1 R$</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>H</td>
<td>$C + e_2 C$</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>O</td>
<td>$H + e_3 H$</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>S</td>
<td>$O + e_4 O$</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

- As we shall see, the mathematical cut-off of division algebras at octonions corresponds to a physical cutoff at 16 component spinors in super Yang-Mills.
To understand the symmetries of the magic square and its relation to YM we shall need in particular two algebras defined on $\mathbb{A}$.

First, the algebra $\text{norm}(\mathbb{A})$ that preserves the norm

$$\langle x|y \rangle := \frac{1}{2}(xy + yx) = x^a y^b \delta_{ab}$$

- $\text{norm}(\mathbb{R}) = 0$
- $\text{norm}(\mathbb{C}) = so(2)$
- $\text{norm}(\mathbb{H}) = so(4)$
- $\text{norm}(\mathbb{O}) = so(8)$
Second, the triality algebra $\text{tri}(\mathbb{A})$

$$\text{tri}(\mathbb{A}) \equiv \{(A, B, C) | A(xy) = B(x)y + xC(y)\}, \ A, B, C \in \mathfrak{so}(n), \ x, y \in \mathbb{A}.$$ 

$$\text{tri}(\mathbb{R}) = 0$$
$$\text{tri}(\mathbb{C}) = \mathfrak{so}(2) + \mathfrak{so}(2)$$
$$\text{tri}(\mathbb{H}) = \mathfrak{so}(3) + \mathfrak{so}(3) + \mathfrak{so}(3)$$
$$\text{tri}(\mathbb{O}) = \mathfrak{so}(8)$$

[Barton and Sudbery:2003]:
3.1 Yang-Mills spin-off: interesting in its own right

- We give a unified description of
  \( D = 3 \) Yang-Mills with \( \mathcal{N} = 1, 2, 4, 8 \)
  \( D = 4 \) Yang-Mills with \( \mathcal{N} = 1, 2, 4 \)
  \( D = 6 \) Yang-Mills with \( \mathcal{N} = 1, 2 \)
  \( D = 10 \) Yang-Mills with \( \mathcal{N} = 1 \)
  in terms of a pair of division algebras \((\mathbb{A}_n, \mathbb{A}_{n\mathcal{N}})\), \( n = D - 2 \)

- We present a master Lagrangian, defined over \( \mathbb{A}_{n\mathcal{N}} \)-valued fields, which encapsulates all cases.

- The overall (spacetime plus internal) on-shell symmetries are given by the corresponding *triality* algebras.

- We use imaginary \( \mathbb{A}_{n\mathcal{N}} \)-valued auxiliary fields to close the non-maximal supersymmetry algebra off-shell. The failure to close off-shell for maximally supersymmetric theories is attributed directly to the non-associativity of the octonions.

[arXiv:1309.0546
A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy]
3.2 $D = 3, \mathcal{N} = 8$ Yang-Mills

- The $D = 3, \mathcal{N} = 8$ super YM action is given by

$$
S = \int d^3 x \left( -\frac{1}{4} F^A_{\mu \nu} F^{A\mu \nu} - \frac{1}{2} D_\mu \phi^A_i D^\mu \phi^A_i + i \bar{\lambda}_a^A \gamma^\mu D_\mu \lambda^A_a \\
- \frac{1}{4} g^2 f_{BC}^A f_{DE}^A \phi_i^B \phi_i^D \phi_j^C \phi_j^E - gf_{BC}^A \phi_i^B \lambda^A_a \Gamma_{ab}^i \lambda^C_b \right),
$$

where the Dirac matrices $\Gamma^i_{ab}, i = 1, \ldots, 7, a, b = 0, \ldots, 7,$ belong to the SO(7) Clifford algebra.

- The key observation is that $\Gamma^i_{ab}$ can be represented by the octonionic structure constants,

$$
\Gamma^i_{ab} = i(\delta_{bi} \delta_{a0} - \delta_{b0} \delta_{ai} + C_{iab}),
$$

which allows us to rewrite the action over octonionic fields.
3.3 Transformation rules

- The supersymmetry transformations in this language are given by

\[
\delta \lambda^A = \frac{1}{2} (F_{A\mu\nu} + \varepsilon^{\mu\nu\rho} D_\rho \phi^A) \sigma_{\mu\nu} \epsilon + \frac{1}{2} g f_{BC} A_i \phi^B \phi^C \sigma_{ij} \epsilon,
\]

\[
\delta A^A_\mu = \frac{i}{2} (\bar{\epsilon} \gamma_\mu \lambda^A - \bar{\lambda}^A \gamma_\mu \epsilon),
\]

\[
\delta \phi^A = \frac{i}{2} e_i [(\bar{\epsilon} e_i) \lambda^A - \bar{\lambda}^A (e_i \epsilon)],
\]

where \(\sigma_{\mu\nu}\) are the generators of \(\text{SL}(2, \mathbb{R}) \cong \text{SO}(1, 2)\).
4.1 Global symmetries of supergravity in D=3

- MATHEMATICS: Division algebras: $R, C, H, O$

  \[(DIVISION\ ALGEBRAS)^2 = MAGIC\ SQUARE\ OF\ LIE\ ALGEBRAS\]

- PHYSICS: $N = 1, 2, 4, 8$ $D = 3$ Yang – Mills

  \[(YANG – MILLS)^2 = MAGIC\ SQUARE\ OF\ SUPERGRAVITIES\]

- CONNECTION: $N = 1, 2, 4, 8 \sim R, C, H, O$

  \[MATHEMATICS\ MAGIC\ SQUARE = PHYSICS\ MAGIC\ SQUARE\]

- The $D = 3$ $G/H$ grav symmetries are given by $ym$ symmetries

  \[G(grav) = \text{tri} \, ym(L) + \text{tri} \, ym(R) + 3[ym(L) \times ym(R)].\]

  eg

  \[E_{8(8)} = SO(8) + SO(8) + 3(\mathbb{1} \times \mathbb{1})\]

  \[248 = 28 + 28 + (8_v, 8_v) + (8_s, 8_s) + (8_c, 8_c)\]
4.2 Squaring $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ Yang-Mills in $D = 3$

Taking a left SYM multiplet

$$\{ A_\mu(L) \in \text{Re}A_L, \quad \phi(L) \in \text{Im}A_L, \quad \lambda(L) \in A_L \}$$

and tensoring with a right multiplet

$$\{ A_\mu(R) \in \text{Re}A_R, \quad \phi(R) \in \text{Im}A_R, \quad \lambda(R) \in A_R \}$$

we obtain the field content of a supergravity theory valued in both $A_L$ and $A_R$:

<table>
<thead>
<tr>
<th>$A_L/A_R$</th>
<th>$A_\mu(R) \in \text{Re}A_R$</th>
<th>$\phi(R) \in \text{Im}A_R$</th>
<th>$\lambda(R) \in A_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_\mu(L) \in \text{Re}A_L$</td>
<td>$g_{\mu\nu} + \phi \in \text{Re}A_L \otimes \text{Re}A_R$</td>
<td>$\varphi \in \text{Re}A_L \otimes \text{Im}A_R$</td>
<td>$\Psi_\mu + \chi \in \text{Re}A_L \otimes A_R$</td>
</tr>
<tr>
<td>$\phi(L) \in \text{Im}A_L$</td>
<td>$\varphi \in \text{Im}A_L \otimes \text{Re}A_R$</td>
<td>$\varphi \in \text{Im}A_L \otimes \text{Im}A_R$</td>
<td>$\chi \in \text{Im}A_L \otimes A_R$</td>
</tr>
<tr>
<td>$\lambda(L) \in A_L$</td>
<td>$\Psi_\mu + \chi \in A_L \otimes \text{Re}A_R$</td>
<td>$\chi \in A_L \otimes \text{Im}A_R$</td>
<td>$\varphi \in A_L \otimes A_R$</td>
</tr>
</tbody>
</table>

Grouping spacetime fields of the same type we find,

$$g_{\mu\nu} \in \mathbb{R}, \quad \Psi_\mu \in \left( \frac{A_L}{A_R} \right), \quad \varphi \in \left( \frac{A_L \otimes A_R}{A_L \otimes A_R} \right), \quad \chi \in \left( \frac{A_L \otimes A_R}{A_L \otimes A_R} \right)$$
4.3 Grouping together

- Grouping spacetime fields of the same type we find,

\[ g_{\mu\nu} \in \mathbb{R}, \quad \psi_\mu \in \left( \frac{A_L}{A_R} \right), \quad \varphi, \chi \in \left( \frac{A_L \otimes A_R}{A_L \otimes A_R} \right). \]  

(3)

- Note we have dualised all resulting p-forms, in particular vectors to scalars. The \( \mathbb{R} \)-valued graviton and \( A_L \oplus A_R \)-valued gravitino carry no degrees of freedom. The \( (A_L \otimes A_R)^2 \)-valued scalar and Majorana spinor each have \( 2(\dim A_L \times \dim A_R) \) degrees of freedom.

- Fortunately, \( A_L \oplus A_R \) and \( (A_L \otimes A_R)^2 \) are precisely the representation spaces of the vector and (conjugate) spinor. For example, in the maximal case of \( A_L, A_R = O \), we have the 16, 128 and 128′ of SO(16).
4.4 Final result

- The $\mathcal{N} > 8$ supergravities in $D = 3$ are unique, all fields belonging to the gravity multiplet, while those with $\mathcal{N} \leq 8$ may be coupled to $k$ additional matter multiplets [Marcus and Schwarz:1983; deWit, Tollsten and Nicolai:1992]. The real miracle is that tensoring left and right YM multiplets yields the field content of $\mathcal{N} = 2, 3, 4, 5, 6, 8$ supergravity with $k = 1, 1, 2, 1, 2, 4$: just the right matter content to produce the U-duality groups appearing in the magic square.
4.5. Conclusion

- In both cases the field content is such that the U-dualities exactly match the groups of the magic square:

<table>
<thead>
<tr>
<th>$\mathbb{A}_L/\mathbb{A}_R$</th>
<th>$\mathbb{R}$</th>
<th>$\mathbb{C}$</th>
<th>$\mathbb{H}$</th>
<th>$\mathbb{O}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>$\text{SL}(2, \mathbb{R})$</td>
<td>$\text{SU}(2, 1)$</td>
<td>$\text{USp}(4, 2)$</td>
<td>$\text{F}_4(-20)$</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td>$\text{SU}(2, 1)$</td>
<td>$\text{SU}(2, 1) \times \text{SU}(2, 1)$</td>
<td>$\text{SU}(4, 2)$</td>
<td>$\text{E}_6(-14)$</td>
</tr>
<tr>
<td>$\mathbb{H}$</td>
<td>$\text{USp}(4, 2)$</td>
<td>$\text{SU}(4, 2)$</td>
<td>$\text{SO}(8, 4)$</td>
<td>$\text{E}_7(-5)$</td>
</tr>
<tr>
<td>$\mathbb{O}$</td>
<td>$\text{F}_4(-20)$</td>
<td>$\text{E}_6(-14)$</td>
<td>$\text{E}_7(-5)$</td>
<td>$\text{E}_8(8)$</td>
</tr>
</tbody>
</table>

Table: Magic square

- The $G/H$ U-duality groups are precisely those of the Freudenthal Magic Square!

\[
G : \quad g_3(\mathbb{A}_L, \mathbb{A}_R) := \text{tri}(\mathbb{A}_L) + \text{tri}(\mathbb{A}_R) + 3(\mathbb{A}_L \times \mathbb{A}_R).
\]

\[
H : \quad g_1(\mathbb{A}_L, \mathbb{A}_R) := \text{tri}(\mathbb{A}_L) + \text{tri}(\mathbb{A}_R) + (\mathbb{A}_L \times \mathbb{A}_R).
\]
4.5a Projective planes?

- U-dualities $G$ are realised non-linearly on the scalars, which parametrise the symmetric spaces $G/H$.
- This can be understood using the remarkable identity relating the projective planes over $(\mathbb{A}_L \otimes \mathbb{A}_R)^2$ to $G/H$,

$$ (\mathbb{A}_L \otimes \mathbb{A}_R)\mathbb{P}^2 \cong G/H. $$

The scalar fields may be regarded as points in division-algebraic projective planes [Baez:2001, Freudenthal:1964, Landsberg2001].

- See also [Atiyah and Berndt:2002].
4.6 Magic Pyramid: G symmetries
We also construct a conformal magic pyramid by tensoring conformal supermultiplets in $D = 3, 4, 6$.

The missing entry in $D = 10$ is suggestive of an exotic theory with $G/H$ duality structure $F_{4(4)}/Sp(3) \times Sp(1)$.
4.8 Conformal Magic Pyramid: G symmetries
5.1 Generalized mirror symmetry: M-theory on $X^7$

We consider compactification of ($\mathcal{N} = 1, D = 11$) supergravity on a 7-manifold $X^7$ with betti numbers $(b_0, b_1, b_2, b_3, b_2, b_1, b_0)$ and define a generalized mirror symmetry

$$(b_0, b_1, b_2, b_3) \rightarrow (b_0, b_1, b_2 - \rho/2, b_3 + \rho/2)$$

under which

$$\rho(X^7) \equiv 7b_0 - 5b_1 + 3b_2 - b_3$$

changes sign

$$\rho \rightarrow -\rho$$

[Duff and Ferrara:2010]

Generalized self-mirror theories are defined to be those for which $\rho = 0$
5.2 Anomalies

<table>
<thead>
<tr>
<th>Field</th>
<th>f</th>
<th>$\Delta A$</th>
<th>$360A$</th>
<th>$360A'$</th>
<th>$X^7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{MN}$</td>
<td>2</td>
<td>-3</td>
<td>848</td>
<td>-232</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>2</td>
<td>0</td>
<td>-52</td>
<td>-52</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>$-b_1 + b_3$</td>
</tr>
<tr>
<td>$\psi_M$</td>
<td>2</td>
<td>1</td>
<td>-233</td>
<td>127</td>
<td>$b_0 + b_1$</td>
</tr>
<tr>
<td>$\chi$</td>
<td>2</td>
<td>0</td>
<td>7</td>
<td>7</td>
<td>$b_2 + b_3$</td>
</tr>
<tr>
<td>$A_{MNP}$</td>
<td>0</td>
<td>2</td>
<td>-720</td>
<td>0</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$A_{\mu\nu\rho}$</td>
<td>1</td>
<td>-1</td>
<td>364</td>
<td>4</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>2</td>
<td>0</td>
<td>-52</td>
<td>-52</td>
<td>$b_2$</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>0</td>
<td>4</td>
<td>4</td>
<td>$b_3$</td>
</tr>
</tbody>
</table>

| total $\Delta A$ | 0       |
| total $A$        | $-\rho/24$ |
| total $A'$       | $-\rho/24$ |

Table: $X^7$ compactification of D=11 supergravity
5.3 Vanish without a trace!

- Remarkably, we find that the anomalous trace depends on $\rho$

\[ A = -\frac{1}{24} \rho(X^7) \]

So the anomaly flips sign under generalized mirror symmetry and vanishes for generalized self-mirror theories.

- Equally remarkable is that we get the same answer for the total trace using the numbers of Grisaru et al 1980.
5.4 Squaring Yang-Mills in $D = 4$ and the self-mirror condition

- Tensoring left and right supersymmetric Yang-Mills theories with field content $(A_\mu, N_L \chi, 2(N_L - 1)\phi)$ and $(A_\mu, N_R \chi, 2(N_R - 1)\phi)$ yields an $N = N_L + N_R$ supergravity theory.

<table>
<thead>
<tr>
<th>$L \setminus R$</th>
<th>$A_\mu$</th>
<th>$N_R \chi$</th>
<th>$2(N_R - 1)\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_\mu$</td>
<td>$g_{\mu\nu} + 2\phi$</td>
<td>$N_R(\psi_\mu + \chi)$</td>
<td>$2(N_R - 1)A_\mu$</td>
</tr>
<tr>
<td>$N_L \chi$</td>
<td>$N_L(\psi_\mu + \chi)$</td>
<td>$N_L N_R(A_\mu + 2\phi)$</td>
<td>$2N_L(N_R - 1)\chi$</td>
</tr>
<tr>
<td>$2(N_L - 1)\phi$</td>
<td>$2(N_L - 1)A_\mu$</td>
<td>$2(N_L - 1)N_R \chi$</td>
<td>$4(N_L - 1)(N_R - 1)\phi$</td>
</tr>
</tbody>
</table>

**Table**: Tensoring $N_L$ and $N_R$ super Yang-Mills theories in $D = 4$. Note that we have dualized the 2-form coming from the vector-vector slot.
5.5 Betti numbers from squaring Yang-Mills

- The betti numbers may then be read off from the Table and we find

\[(b_0, b_1, b_2, b_3) = (1, N_L + N_R - 1, N_L N_R + N_L + N_R - 3, 3N_L N_R - 2N_L - 2N_R + 3)\]

Consequently

\[\rho (X^7) = 7b_0 - 5b_1 + 3b_2 - b_3 = 0\]  \hspace{1cm} (4)

- Similar results hold in \(D = 5\) where

\[(c_0, c_1, c_2, c_3) = (1, N_L + N_R - 2, N_L N_R - 1, 2N_L N_R - 2N_L - 2N_R + 4)\]

Consequently

\[\chi (X^6) = 2b_0 - 2c_1 + 2c_2 - c_3 = 0\]  \hspace{1cm} (5)