Disentangling mass and angle dependance in neutrino mixing

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Outline

- Flavor mixing in Quantum Field Theory (QFT)
- Decomposition of the mixing generator
- Conclusions

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Neutrino mixing in Quantum Field Theory (QFT)

Mixing relations for two Dirac fields

\[ \nu_e(x) = \cos \theta \, \nu_1(x) + \sin \theta \, \nu_2(x) \]

\[ \nu_\mu(x) = -\sin \theta \, \nu_1(x) + \cos \theta \, \nu_2(x) \]

\( \nu_1, \nu_2 \) are fields with definite masses.

Mixing transformations connect the two quadratic forms:

\[ \mathcal{L} = \bar{\nu}_1 (i \, \not\! \partial - m_1) \nu_1 + \bar{\nu}_2 (i \, \not\! \partial - m_2) \nu_2 \]

and

\[ \mathcal{L} = \bar{\nu}_e (i \, \not\! \partial - m_e) \nu_e + \bar{\nu}_\mu (i \, \not\! \partial - m_\mu) \nu_\mu - m_{e\mu} \left( \bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e \right) \]

with \( m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta \), \( m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta \), \( m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta \).
Neutrino mixing in Quantum Field Theory (QFT)

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with \( m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta \), \( m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta \), \( m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta \).
Mixing relations can be written as:²

\[ \nu_{e}^{\alpha}(x) = G_{\theta}^{-1}(t) \nu_{1}^{\alpha}(x) G_{\theta}(t) \]

\[ \nu_{\mu}^{\alpha}(x) = G_{\theta}^{-1}(t) \nu_{2}^{\alpha}(x) G_{\theta}(t) \]

with the mixing generator³

\[ G_{\theta}(t) = \exp \left[ \theta \int d^{3}x \left( \nu_{1}^{\dagger}(x)\nu_{2}(x) - \nu_{2}^{\dagger}(x)\nu_{1}(x) \right) \right] \]

---

³ For \( \nu_{e} \), we get \( \frac{d^2}{d\theta^2} \nu_{e}^{\alpha} = -\nu_{e}^{\alpha} \) with initial conditions

\[ \nu_{e}^{\alpha} \big|_{\theta=0} = \nu_{1}^{\alpha} , \quad \frac{d}{d\theta} \nu_{e}^{\alpha} \big|_{\theta=0} = \nu_{2}^{\alpha} \].

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Disentangling mass and angle dependance in neutrino mixing
Where $\nu_i$ are free Dirac field operators:

$$\nu_i(x) = \sum_{k,r} \frac{e^{ik \cdot x}}{\sqrt{V}} \left[ u^r_{k,i}(t) \alpha^r_{k,i} + v^r_{-k,i}(t) \beta^r_{-k,i} \right], \quad i = 1, 2.$$  

Anticommutation, orthonormality and completeness relations are the standard ones.
The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t) |0\rangle_{1,2}$$

Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V \to \infty} \langle 0|0(t)\rangle_{e,\mu} = \lim_{V \to \infty} e^V \int \frac{d^3k}{(2\pi)^3} \ln \left(1-\sin^2 \theta |V_k|^2\right)^2 = 0$$

with

$$|V_k|^2 \equiv \sum_{r,s} |v_{-k,1}^{r\dagger} u_{k,2}^s|^2 \neq 0 \quad \text{for} \quad m_1 \neq m_2$$
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Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V \to \infty} 1,2 \langle 0 | 0(t) \rangle_{e,\mu} = \lim_{V \to \infty} \left( V \int \frac{d^3k}{(2\pi)^3} \ln \left( 1 - \sin^2 \theta |V_k|^2 \right)^2 \right) = 0$$

with

$$|V_k|^2 \equiv \sum_{r,s} \left| v^{r\dagger}_{-k,1} u^s_{k,2} \right|^2 \neq 0 \quad \text{for} \quad m_1 \neq m_2$$

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Condensation density:
\[ e,\mu \langle 0(t)|\alpha_{k,i}^{r\dagger}\alpha_{k,i}^{r}|0(t)\rangle e,\mu = e,\mu \langle 0(t)|\beta_{k,i}^{r\dagger}\beta_{k,i}^{r}|0(t)\rangle e,\mu = \sin^2 \theta |V_k|^2 \]

vanishing for \( m_1 = m_2 \) and/or \( \theta = 0 \) (in both cases no mixing).
Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$
\alpha_{k,e}^r(t) = \cos \theta \alpha_{k,1}^r + \sin \theta \left( U_k^*(t) \alpha_{k,2}^r + \epsilon^r V_k(t) \beta_{-k,2}^{r\dagger} \right)
$$

$$
\alpha_{k,\mu}^r(t) = \cos \theta \alpha_{k,2}^r - \sin \theta \left( U_k(t) \alpha_{k,1}^r - \epsilon^r V_k(t) \beta_{-k,1}^{r\dagger} \right)
$$

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$$

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$$

Mixing transformation = Rotation nested with a Bogoliubov transformation\(^4\).

\[^4\]U_k(t) = u_{k,2}^{r\dagger} u_{k,1}^r e^{i(\omega_{k,2} - \omega_{k,1})t}; |U_k|^2 + |V_k|^2 = 1
Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

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$$\beta_{-k,e}^r(t) = \cos \theta \beta_{-k,1}^r + \sin \theta \left( U_k^*(t) \beta_{-k,2}^r - \epsilon^r V_k(t) \alpha_{k,2}^{r\dagger} \right)$$

$$\beta_{-k,\mu}^r(t) = \cos \theta \beta_{-k,2}^r - \sin \theta \left( U_k(t) \beta_{-k,1}^r + \epsilon^r V_k(t) \alpha_{k,1}^{r\dagger} \right)$$

Mixing transformation = Rotation nested with a Bogoliubov transformation$^4$.

\[4U_k(t) = u_{k,2}^{r\dagger} u_{k,1}^r e^{i(\omega_k,2 - \omega_k,1)t}; |U_k|^2 + |V_k|^2 = 1\]
Neutrino oscillation formula (exact result)\textsuperscript{5}:

\[
Q_{k,\nu_e}(t) = 1 - |U_k|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t\right) \\
- |V_k|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t\right)
\]

\[
Q_{k,\nu_\mu}(t) = |U_k|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2} t\right) \\
+ |V_k|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2} t\right)
\]

For \( k \gg \sqrt{m_1 m_2} \), \( |U_k|^2 \to 1 \) and \( |V_k|^2 \to 0 \).

Decomposition of mixing generator

Mixing generator function of $m_1$, $m_2$, and $\theta$.

$$G_\theta(t) = \exp \left[ \theta \int d^3 x \left( \nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x) \right) \right]$$

$$= \exp \left\{ \sum_r \left( U^*_k \alpha^r_{k,1} \alpha^r_{k,2} - \epsilon^r V^*_k \beta^r_{-k,1} \alpha^r_{k,2} + \epsilon^r V_k \alpha^{r\dagger}_{k,1} \beta^{r\dagger}_{-k,2} + U_k \beta^r_{-k,1} \beta^{r\dagger}_{-k,2} \right) \right\}$$

Our aim is to disentangle the mass dependence from the one by the mixing angle.
Decomposition of mixing generator

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\[
G_\theta(t) = \exp \left[ \theta \int d^3 x \left( \nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right] \\
= \exp \left\{ \sum_r \left( U_k^* \alpha_{k,1}^r \alpha_{k,2}^r - \epsilon^r V_k \beta_{-k,1}^r \alpha_{k,2}^r + \epsilon^r V_k \alpha_{k,1}^r \beta_{-k,2}^r + U_k \beta_{-k,1}^r \beta_{-k,2}^r \right) \right\} \\
- \sum_r \left( U_k \alpha_{k,2}^{r\dagger} \alpha_{k,1}^r + \epsilon^r V_k^* \beta_{-k,2}^r \alpha_{k,1}^r - \epsilon^r V_k \alpha_{k,2}^{r\dagger} \beta_{-k,1}^r + U_k^* \beta_{-k,2}^r \beta_{-k,1}^r \right) \right\}
\]

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Mixing generator function of $m_1$, $m_2$, and $\theta$.

\[
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\]

\[
= \exp \left\{ \sum_r \left( U^*_k \alpha_{k,1}^r \alpha_{k,2}^r - \epsilon^r V_k^* \beta_{-k,1}^r \alpha_{k,2}^r + \epsilon^r V_k \alpha_{k,1}^r \beta_{-k,2}^r + U_k \beta_{-k,1}^r \beta_{-k,2}^r \right) \right\}
\]

Our aim is to disentangle the mass dependence from the one by the mixing angle.
Let us define\textsuperscript{6}:

\[
R(\theta) \equiv \exp \left\{ \theta \sum_{k,r} \left[ (\alpha_{k,1}^r \alpha_{k,2}^r + \beta_{-k,1}^r \beta_{-k,2}^r) e^{i\psi_k} + \right.ight.
\left. \left. - (\alpha_{k,2}^r \alpha_{k,1}^r + \beta_{-k,2}^r \beta_{-k,1}^r) e^{-i\psi_k} \right] \right\},
\]

\[
B_i(\Theta_i) \equiv \exp \left\{ \sum_{k,r} \Theta_{k,i} \epsilon_r \left[ \alpha_{k,i}^r \beta_{-k,i}^r e^{-i\phi_{k,i}} - \beta_{-k,i}^r \alpha_{k,i}^r e^{i\phi_{k,i}} \right] \right\},
\]

\[i = 1, 2\]

Since \([B_1, B_2] = 0\) we put \(B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1) B_2(\Theta_2)\)

\textsuperscript{6}

\[
\psi_k = (\omega_{k,1} - \omega_{k,2}) t; \quad \phi_{k,i} = 2 \omega_{k,i} t; \quad \Theta_{k,i} = \frac{1}{2} \cot^{-1} \left( \frac{|k|}{m_i} \right)
\]
The \( B_i(\Theta_{k,i}) \), \( i = 1, 2 \) are ordinary Bogoliubov transformations which introduce a mass shift, and \( R(\theta) \) is a rotation.

Their action on the vacuum is given by:

\[
|0\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} = \prod_{k,r} \left[ \cos \Theta_{k,i} + \epsilon^r \sin \Theta_{k,i} \alpha_{k,i}^{r\dagger} \beta_{-k,i}^{r\dagger} \right] |0\rangle_{1,2}
\]

\[
R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}.
\]
The $B_i(\Theta_{k,i}), i = 1, 2$ are ordinary Bogoliubov transformations which introduce a mass shift, and $R(\theta)$ is a rotation.

Their action on the vacuum is given by:

$$|\tilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} = \prod_{k,r} \left[ \cos \Theta_{k,i} + \epsilon^r \sin \Theta_{k,i} \alpha_{k,i}^r \beta_{-k,i}^r \right] |0\rangle_{1,2}$$

$$R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}.$$
We find:

\[ G_\theta = B(\Theta_1, \Theta_2) \ R(\theta) \ B^{-1}(\Theta_1, \Theta_2) \]

which is realized when the \( \Theta_{k,i} \) are chosen as:

\[
U_k = e^{-i\psi_k} \cos(\Theta_{k,1} - \Theta_{k,2}) \\
V_k = e^{\frac{(\phi_{k,1} + \phi_{k,2})}{2}} \sin(\Theta_{k,1} - \Theta_{k,2})
\]
Then we rewrite the generator as

\[ G(\theta, m_1, m_2) = B^{-1}(m_1, m_2) R(\theta) B(m_1, m_2) \]

Moreover,

\[ |0\rangle_{e,\mu} = G^{-1}|0\rangle_{1,2} = |0\rangle_{1,2} + \left[ B(m_1, m_2) , R^{-1}(\theta) \right] |\tilde{0}\rangle_{1,2} \]

which shows that the condensate structure of the flavor vacuum arises as a consequence of the non vanishing commutator \([B, R^{-1}]\).
Then we rewrite the generator as

$$G(\theta, m_1, m_2) = B^{-1}(m_1, m_2) R(\theta) B(m_1, m_2)$$

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In the following table we report the vacuum expectation values of the (unordered) Hamiltonian on the various *vacua* obtained by acting step-by-step with the above generators.

<table>
<thead>
<tr>
<th>( \langle H_{k,1} + H_{k,2} \rangle )</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>(- (\omega_{k,1} + \omega_{k,2}) )</td>
<td>(</td>
</tr>
<tr>
<td>(- (k + k) )</td>
<td>( B^{-1}(m_1, m_2)</td>
</tr>
<tr>
<td>(- k(2 \cos^2 \theta + \frac{\omega_{k,1}}{\omega_{k,2}} \frac{\omega_{k,2}}{\omega_{k,1}} \sin^2 \theta) )</td>
<td>( R^{-1}(\theta)B^{-1}(m_1, m_2)</td>
</tr>
<tr>
<td>(- (\omega_{k,1} + \omega_{k,2})(1 - 2 \sin^2 \theta \sin^2(\Theta_{k,1} - \Theta_{k,2})) )</td>
<td>( B(m_1, m_2)R^{-1}(\theta)B^{-1}(m_1, m_2)</td>
</tr>
</tbody>
</table>
Plot of vacuum energies for $H_1$ and $H_2$ for the different vacua given in Table for

$$\theta = \pi/6, \ m_1 = 20, \ m_2 = 150, \ k = 80.$$
Plot of vevs of $H_1$ and $H_2$ for $\theta = \pi/4$, $k = 300$, $m_1 = 20$, $m_2 = 150$.

Plot of vevs of $H_1$ and $H_2$ for $\theta = \pi/10$, $k = 80$, $m_1 = 20$, $m_2 = 150$.

Plot of vevs of $H_1$ and $H_2$ for $\theta = \pi/4$, $k = 0$, $m_1 = 20$, $m_2 = 20$.
Conclusion

- Mixing transformations are not trivial in Q.F.T. ⇔ they are associated to inequivalent representations.
- The mixing Generator can be seen as a rotation transformed under a Bogoliubov transformation.
- The seed of the inequivalence between the two vacua (mass and flavor) is in the non commutativity between the rotation and the Bogoliubov transformation.
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work in progress: Can we consider a "thermodynamical" interpretation of the energies associated to the different vacua (mass and flavour)?
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work in progress: Can we consider a ”thermodynamical” interpretation of the energies associated to the different vacua (mass and flavour)?
Thank you for your kind attention
According to Thermal Field Theory (TFT) a state such as 
\[ |0(\vartheta)\rangle_i = B(\vartheta) |0\rangle \] can be written as
\[ |0(\vartheta)\rangle = \exp\left(-\frac{S_\alpha}{2}\right) |I\rangle = \exp\left(-\frac{S_\beta}{2}\right) |I\rangle \]

with
\[ |I\rangle \equiv \exp \left( \sum_{k,r} (-1)^{i+1} \epsilon r e^{i\phi} \beta_r^{\dagger} \alpha_k^{\dagger} \right) |0\rangle \]
\[ S_\alpha = - \sum_{k,r} \left( \alpha^r_k \alpha^r_k \log \sin^2 \vartheta_k + \alpha^r_k \alpha^r_k \log \cos^2 \vartheta_k \right) \]

A similar expression holds for \( S_\beta \). It is known\(^7\) that \( S_\alpha \) (or \( S_\beta \)) can be interpreted as the entropy function associated to the vacuum condensate.

With

\[ n_k = \langle 0(\vartheta) | \alpha^r_k \alpha^r_k | 0(\vartheta) \rangle = \sin^2(\vartheta_k) \]

we have that the expectation value of the entropy operator is

\[ S_k = \langle 0(\vartheta) | S_\alpha | 0(\vartheta) \rangle = n_k \log(n_k) + (1 - n_k) \log(1 - n_k) \]

At this point it is natural to write a formally analogue expression for $S_{ik}$, entropy associated to the state we obtain after the first Bogoliubov transformation and $S_{\sigma k}$, entropy associated to the last Bogoliubov transformation.

$$S_{\sigma k} = n_{\sigma k} \log(n_{\sigma k}) + (1 - n_{\sigma k}) \log(1 - n_{\sigma k})$$

with

$$n_{\sigma k} = \sin^2 \theta \sin^2(\Theta_{1k} - \Theta_{2k}) \quad \sigma = e, \mu$$

being the expectation value of the number operator on the flavor vacuum.
We thus obtain

\[ S_{ik} = \sin^2(\Theta_{ik}) \log(\sin^2(\Theta_{ik}) + \cos^2(\Theta_{ik}) \log(\cos^2(\Theta_{ik}))) \]
\[ S_{\sigma k} = \sin^2 \theta \sin^2(\Theta_{1k} - \Theta_{2k}) \log(\sin^2 \theta \sin^2(\Theta_{1k} - \Theta_{2k})) + \]
\[ + (1 - \sin^2 \theta \sin^2(\Theta_{1k} - \Theta_{2k})) \log(1 - \sin^2 \theta \sin^2(\Theta_{1k} - \Theta_{2k})) \]

which means that, as we expected, the two states have different entropies.
Motivation

- CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;
- Theoretical interest: origin of mixing in the Standard Model;
- Bargmann superselection rule: coherent superposition of states with different masses is not allowed in non-relativistic QM;
- Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles; oscillation formulas;

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Anticommutation, orthonormality and completeness relations

Anticommutation relations:

\[ \{ \nu_i^\alpha (x), \nu_j^{\beta \dagger} (y) \}_{t=t'} = \delta^3 (x-y) \delta_{\alpha \beta} \delta_{ij} \]
\[ \{ \alpha_r^{k,i} \, , \alpha_{q,j}^{s \dagger} \} = \{ \beta_r^{k,i} \, , \beta_{q,j}^{s \dagger} \} = \delta^3 (k-q) \delta_{rs} \delta_{ij} \]

Orthonormality and completeness relations:

\[ u^r_{k,i} (t) = e^{-i \omega_{k,i} t} u^r_{k,i} \; ; \; \; v^r_{k,i} (t) = e^{i \omega_{k,i} t} v^r_{k,i} \; ; \; \; \omega_{k,i} = \sqrt{k^2 + m_i^2} \]

\[ u^r_{k,i} u^{s \dagger}_{k,i} = v^r_{k,i} v^{s \dagger}_{k,i} = \delta_{rs} \; , \; \; u^r_{k,i} v^{s \dagger}_{-k,i} = 0 \; , \; \; \sum_r (u^r_{k,i} u^{r \dagger}_{k,i} + v^r_{-k,i} v^{r \dagger}_{-k,i}) = \delta_{\alpha \beta} \cdot \]

Fock space for \( \nu_1, \nu_2 \):

\[ \mathcal{H}_{1,2} = \{ \alpha_{1,2}^\dagger , \beta_{1,2}^\dagger \, , \, |0\rangle_{1,2} \} . \]
Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:
  - finite \# of degrees of freedom
  - unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).*

- Quantum Field Theory:
  - infinite \# of degrees of freedom.
  - \( \infty \) many unitarily inequivalent representations of the field algebra \( \Leftrightarrow \) many vacua.
  - The mapping between interacting and free fields is “weak”, i.e. representation dependent (LSZ formalism)\(^{11}\). Example: theories with spontaneous symmetry breaking.

Condensation density for mixed fermions

- $V_k = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \rightarrow \frac{1}{2}$ for
  $\frac{(m_2 - m_1)^2}{m_1 m_2} \rightarrow \infty$.
- $|V_k|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$. 
In definitive, the flavor fields can be expanded as \(((\sigma, i) = (e, 1), (\mu, 2))\):

\[
\nu_{\sigma}(x) = \sum_{r=1,2} \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} \left[ u_{k,i}^r(t) \alpha_{k,\sigma}^r(t) + v_{-k,i}^r(t) \beta_{-k,\sigma}^r(t) \right] e^{i k \cdot x},
\]

with

\[
\alpha_{k,\sigma}^r(t) = G^{-1}_\theta(t) \alpha_{k,i}^r G_\theta(t)
\]

\[
\beta_{k,\sigma}^r(t) = G^{-1}_\theta(t) \beta_{k,i}^r G_\theta(t).
\]

The flavor ladder operators can be also obtained by comparing the field expansions and using:

\[
\alpha_{k,\sigma}^r(t) = \int d^3 x \; u_{k,i}^{r\dagger}(x) \nu_{\sigma}(x),
\]

\[
\beta_{-k,\sigma}^{r\dagger}(t) = \int d^3 x \; v_{-k,i}^{r\dagger}(x) \nu_{\sigma}(x).
\]
The “flavor vacuum” $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state$^{12}$:

$$|0\rangle_{e,\mu} = \prod_{k,r} \left[ (1 - \sin^2 \theta |V_k|^2) - \epsilon^r \sin \theta \cos |V_k| (\alpha_{r\downarrow}^r \beta_{r\downarrow}^r - \alpha_{r\uparrow}^r \beta_{r\uparrow}^r) \right. \right.$$  

$$+ \epsilon^r \sin^2 \theta |V_k||U_k| (\alpha_{r\downarrow}^r \beta_{r\downarrow}^r - \alpha_{r\uparrow}^r \beta_{r\uparrow}^r) + \sin^2 \theta |V_k|^2 (\alpha_{r\downarrow}^r \beta_{r\downarrow}^r - \alpha_{r\uparrow}^r \beta_{r\uparrow}^r) \left|0\right\rangle_{1,2}$$

$^{12}$ A. Perelomov, *Generalized Coherent States and Their Applications*, (Springer V., 1986)
Lagrangian in the mass basis:

\[ \mathcal{L} = \bar{\nu}_m (i \partial - M_d) \nu_m \]

where \( \nu^T_m = (\nu_1, \nu_2) \) and \( M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \).

\[ \text{---} \]

\[ ^{13}\text{M. Blasone, P. Jizba and G. Vitiello, Phys. Lett. B (2001)} \]
$\mathcal{L}$ invariant under global $U(1)$ with conserved charge $Q = \text{total charge}$.
Consider now the $SU(2)$ transformation:

$$\nu'_m = e^{i\alpha_j \tau_j} \nu_m \quad ; \quad j = 1, 2, 3.$$ 

with $\tau_j = \sigma_j / 2$ and $\sigma_j$ being the Pauli matrices.
The associated currents are:

\[
\delta \mathcal{L} = i \alpha_j \bar{\nu}_m [\tau_j, M_d] \nu_m = -\alpha_j \partial_\mu J^\mu_{m,j} 
\]

\[
J^\mu_{m,j} = \bar{\nu}_m \gamma^\mu \tau_j \nu_m 
\]

The charges \( Q_{m,j}(t) \equiv \int d^3x \, J^0_{m,j}(x) \), satisfy the \( su(2) \) algebra:

\[
[Q_{m,j}(t), Q_{m,k}(t)] = i \epsilon_{jkl} Q_{m,l}(t). 
\]
The Casimir operator is proportional to the total charge: 
\[ C_m = \frac{1}{2} Q. \]

\[ Q_{m,3} \] is conserved \( \Rightarrow \) charge conserved separately for \( \nu_1 \) and \( \nu_2 \):

\[ Q_1 = \frac{1}{2} Q + Q_{m,3} = \int d^3x \, \nu_1^\dagger(x) \nu_1(x) \]

\[ Q_2 = \frac{1}{2} Q - Q_{m,3} = \int d^3x \, \nu_2^\dagger(x) \nu_2(x). \]

These are the flavor charges in the absence of mixing.
The currents in the flavor basis

Lagrangian in the flavor basis:

\[ \mathcal{L} = \bar{\nu}_f (i \not{\partial} - M) \nu_f \]

where \( \nu_f^T = (\nu_e, \nu_\mu) \) and \( M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix} \).

Consider the \( SU(2) \) transformation:

\[ \nu'_f = e^{i\alpha_j \tau_j} \nu_f \quad ; \quad j = 1, 2, 3. \]

with \( \tau_j = \sigma_j / 2 \) and \( \sigma_j \) being the Pauli matrices.
The charges $Q_{f,j} \equiv \int d^3x J_{f,j}^0$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i \epsilon_{jkl} Q_{f,l}(t).$$

0.2cm
The Casimir operator is proportional to the total charge $C_f = C_m = \frac{1}{2} Q$. 
• $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between $\nu_e$ and $\nu_\mu$.

Define the flavor charges as:

$$Q_e(t) \equiv \frac{1}{2} Q + Q_{f,3}(t) = \int d^3x \, \nu_e^\dagger(x) \nu_e(x)$$

$$Q_\mu(t) \equiv \frac{1}{2} Q - Q_{f,3}(t) = \int d^3x \, \nu_\mu^\dagger(x) \nu_\mu(x)$$

where $Q_e(t) + Q_\mu(t) = Q$. 
We have:

\[
Q_e(t) = \cos^2 \theta Q_1 + \sin^2 \theta Q_2 + \sin \theta \cos \theta \int d^3x \left[ \nu^\dagger_1 \nu_2 + \nu^\dagger_2 \nu_1 \right]
\]

\[
Q_\mu(t) = \sin^2 \theta Q_1 + \cos^2 \theta Q_2 - \sin \theta \cos \theta \int d^3x \left[ \nu^\dagger_1 \nu_2 + \nu^\dagger_2 \nu_1 \right]
\]
In conclusion:

- In presence of mixing, neutrino flavor charges are defined as

\[ Q_e(t) \equiv \int d^3x \, \nu_e^{\dagger}(x) \nu_e(x) \; ; \; \quad Q_\mu(t) \equiv \int d^3x \, \nu_\mu^{\dagger}(x) \nu_\mu(x) \]

- They are not conserved charges \( \Rightarrow \) flavor oscillations.
- They are still (approximately) conserved in the vertex \( \Rightarrow \) define flavor neutrinos as their eigenstates
  - Problem: find the eigenstates of the above charges.
The flavor charge operators are diagonal in the flavor ladder operators:

\[ :: Q_{\nu\sigma}(t) :: \equiv \int d^3x :: \nu_\sigma^\dagger(x) \nu_\sigma(x) :: \]

\[ = \sum_r \int d^3k \left( \alpha_{k,\sigma}^r(t) \alpha_{k,\sigma}^r(t) - \beta_{-k,\sigma}^r(t) \beta_{-k,\sigma}^r(t) \right), \quad \sigma = e, \mu. \]

Here \( :: ... :: \) denotes normal ordering with respect to the flavor vacuum:

\[ :: A :: \equiv A - e,\mu \langle 0 | A | 0 \rangle_{e,\mu} \]

- Define flavor neutrino states with definite momentum and helicity:

\[ | \nu_{k,\sigma}^r \rangle \equiv \alpha_{k,\sigma}^r(0) | 0 \rangle_{e,\mu} \]

- Such states are eigenstates of the flavor charges (at \( t=0 \)):

\[ :: Q_{\nu\sigma} :: | \nu_{k,\sigma}^r \rangle = | \nu_{k,\sigma}^r \rangle \]
We have, for an electron neutrino state:

\[ Q_{k,\nu_\sigma}(t) \equiv \langle \nu_{k,e}^r | : Q_{\nu_\sigma} (t) : | \nu_{k,e}^r \rangle \]

\[
= \left| \{ \alpha_{k,\sigma}^r (t), \alpha_{k,e}^r (0) \} \right|^2 + \left| \{ \beta_{-k,\sigma}^r (t), \alpha_{k,e}^r (0) \} \right|^2
\]
Neutrino oscillation formula (exact result)

\[ Q_{k, \nu_e}(t) = 1 - |U_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |V_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \]

\[ Q_{k, \nu_\mu}(t) = |U_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \]

- For \( k \gg \sqrt{m_1 m_2} \), \( |U_k|^2 \rightarrow 1 \) and \( |V_k|^2 \rightarrow 0 \).

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