In what follows use of the above constraint, the system can be eventually reduced to a three-dimensional dynamical system in the new variables

\[ \Lambda \mu \nu = \dot{\Lambda}^2 + A(t)^2 \frac{d\theta^2}{\Lambda^2} + B(t)^2 \sin^2 \theta d\phi^2. \]

The global structure of these models was described by Collins, who was also the first who analyzed the model as a two-dimensional dynamical system for the case of perfect fluid with vanishing cosmological constant. Motivated by a series of recent papers concerning the analyses of self-gravitating Skyrme fields in cosmology and Kantowski-Sachs spacetimes, we consider Kantowski-Sachs cosmological models sourced by a Skyrme field and a cosmological constant in the framework of General Relativity.

### Einstein-Skyrme system

The Skyrme model is a generalized nonlinear sigma model. Although not involving quarks, it can be regarded as an approximate, low energy effective theory of QCD, whose topological soliton solutions can be interpreted as baryons (Skyrmions).

The Einstein-Skyrme equations read:

\[ G_{\mu\nu} + \Lambda_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1} \]

\[ \nabla^\mu R_{\mu\nu} + \frac{2}{3} \nabla^\mu (\nabla \Phi \cdot \nabla F_{\mu\nu}) = 0 \tag{2} \]

where \( T_{\mu\nu} \) is the energy-momentum tensor for the Skyrme field, \( \Lambda_{\mu\nu} \) is the Einstein tensor, \( G \) is the cosmological constant, \( \Phi \) is the Skyrme field, \( R_{\mu\nu} \) is the Riemann tensor, and \( F_{\mu\nu} \) is the electromagnetic field tensor.

\( \Lambda_{\mu\nu} \) is the cosmological constant, \( \Phi \) is the Skyrme field, and \( R_{\mu\nu} \) is the Riemann tensor.

In what follows, besides the choice of Kantowski-Sachs spacetimes, we consider the particular case of a constant radial profile function \( a = \pi R \), so that Eq.(2), reduced to scalar equation by the hedgehog ansatz, is identically solved. Eq.(1) can be further manipulated and written in terms of propagation equations for the usual volume expansion scalar \( \theta \), the shear scalar \( \varphi = (1/2) \epsilon_{\mu\nu} \partial_{\mu} \pi_{\nu} \), and the 3-curvature scalar \( \partial R \).

\[ \dot{\theta} + \frac{1}{2} \partial^2 + 2\sigma^2 = \frac{\Lambda - 3k\varphi^2}{8} \]
\[ \dot{\sigma} + \theta - \frac{1}{2\sqrt{3}} \varphi^2 = -\frac{k\sigma R}{4\sqrt{3}} \]
\[ \dot{R} + \frac{3}{2} \partial^2 - 2\varphi^2 = 2\Lambda + k\sigma R \left( 1 + \frac{\lambda R^2}{4} \right). \]

In what follows \( \varphi \) is fixed and we will consider \( \theta < \epsilon \).

### Dynamical system

Introducing the normalization function

\[ Q = \frac{y}{\sqrt{9} \sqrt{2}}, \quad S = \frac{y^2}{\sqrt{2}}, \quad \Omega_{\lambda} = \frac{x}{\sqrt{3}}, \quad \Omega_{\phi} = \frac{y}{\sqrt{6}}, \quad \Omega_{\chi} = \frac{k\lambda^2 R^2}{24x^2} \]

allows to construct a compact state space since the constraint in Eq.(6) becomes

\[ Q^2 + \Omega_{\lambda} = 1, \quad \Omega_{\phi} + S^2 + \chi(1 - Q^2) = \Omega_{\chi} + 1. \]

Differenitiating with respect to the new time variable \( \tau = (1/D) dt \) and making use of the above constraint, the system can be eventually reduced to a three-dimensional dynamical system in the new variables \( Q, \Sigma, \Omega_{\lambda} : \)

\[ Q' = (Q^2 - 1)(1 - k(1 - Q^2) + Q \Sigma + S^2 - 2\Omega_{\lambda}) \]
\[ \Sigma' = k[1 - Q^2 - (1 - 3\Sigma^2) + Q \Sigma + S^2 + 2(1 - Q^2) \Omega_{\lambda}] \]
\[ \Omega_{\lambda}' = 2(Q(2(1 - Q^2) + Q(1 - S^2) - 2\Omega_{\lambda}) \Omega_{\chi} \]

with compact phase space

\[ S = (Q, \Sigma, \Omega_{\lambda}), \quad \Omega_{\phi} \epsilon [-1 < Q < 1, -1 < \Sigma \leq 1, 0 \leq \Omega_{\lambda} \leq 1, 0 < \Omega_{\phi} < 2(1 - Q^2), 0 < \Omega_{\chi} < 1] \]

The system admits six stationary points listed below.

<table>
<thead>
<tr>
<th>Point</th>
<th>( Q )</th>
<th>( \Sigma )</th>
<th>( \Omega_{\lambda} )</th>
<th>( \Omega_{\phi} )</th>
<th>( \Omega_{\chi} )</th>
<th>Stability</th>
<th>( q )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Stable</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>B</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>Unstable</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>C</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Stable</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Stable</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>G</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>Unstable</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Unstable</td>
<td>2</td>
<td>-1</td>
</tr>
</tbody>
</table>

It also displays a normally hyperbolic equilibrium set in the \( Q - \Sigma \) plane, defined by:

\[ \sqrt{\frac{1-k}{1-k}} \leq Q \leq \sqrt{\frac{1-k}{1-k}}, \quad \Omega_{\lambda} = \frac{1}{2}(1 + 2Q^2 + k(Q^2 - 1)) \]

and three invariant submanifolds characterized by \( Q = \pm 1 \), \( Q = -1 \) and \( \Omega_{\lambda} = 0 \) respectively.

### Discussion

The numerical value of the deceleration parameter \( q \) and the analytic behavior of the expansion scalar \( \theta \) at each fixed point can be easily evaluated. Two classes of solution are obtained:

- The stationary points A, C, F and H are characterized by \( Q = \pm 1 \) and \( \Sigma = \pm 1 \); this allows to solve in terms of both scale factors \( A \) and \( B \) to obtain either \( B \sim 0 \) and \( A \sim t \), or \( B \sim t^{2/3} \) and \( A \sim t^{1/3} \) depending on the sign of \( \Sigma \). These solutions are anisotropic and display Kasner-like behaviours.
- The solutions represented by the stationary points B and G have vanishing shear; they are said to undergo isotropization which, in this context, means that the two scale factors are characterized by the same functional dependence on time. They are driven by the cosmological constant, the sign of the exponent depending on the sign of \( Q \):

\[ B \sim A \sim e^{\sqrt{\frac{3}{t}}}, \]

thus these solutions are said de Sitter-like solutions. Analogously, for the equilibrium set the acceleration parameter is always \( q = -1 \).

A bounce occurs on one of the scale factors, say \( A \), is said to occur at time \( t^* \) if and only if \( \dot{y}(t^*) = 0 \) and \( d\dot{y} \neq 0 \), where \( y(1/A) d\dot{y} \).

It can be easily shown that, in this model, a bounce in the scale factor \( B \) is impossible while a bounce in the scale factor \( A \) requires the violation of the Strong Energy Condition of the total matter-energy content.

### Acknowledgments

This work is partially supported by Agenzia Spaziale Italiana (ASI) through Contract No. 1/304/12/0. The authors acknowledge support by Istituto Nazionale di Fisica Nucleare (INFN) and by the Italian Ministero dell’Istruzione, dell’Università e della Ricerca (MIUR).

### References