Rotation, Bogoliubov transformation and neutrino mixing

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Introduction

We show that mixing transformations for Dirac fields arise as a consequence of the non-trivial interplay between rotations and Bogoliubov transformations at level of ladder operators. Indeed the non-commutativity between the algebraic generators of such transformations turns out to be responsible of the unitary inequivalence of the flavor and mass representations and of the associated vacuum structure. A possible thermodynamic interpretation is also investigated.

Pontecorvo vs QFT

Pontecorvo mixing transformations are written as a rotation of the states with definite masses $|\nu_1\rangle$, $|\nu_2\rangle$, into those with definite flavor $|\nu_e\rangle$ and $|\nu_\mu\rangle$ as [1]

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle, \tag{1}$$
$$|\nu_\mu\rangle = \cos\theta\,|\nu_2\rangle - \sin\theta\,|\nu_1\rangle. \tag{2}$$

On the other hand, Standard Model is formulated in terms of fields[1] and there neutrino mixing appears in the following form [2]

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)\,, \tag{3}$$
$$\nu_\mu(x) = \cos\theta\,\nu_2(x) - \sin\theta\,\nu_1(x)\,, \tag{4}$$

where $x \equiv (\mathbf{x}, t)$. The generator of such a transformation is [3]

$$G(t;\theta,m_1,m_2) = \exp\left\{\theta\int d^3\mathbf{x}\left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right\}\,. \tag{5}$$

The question then arise to what extent the two above transformations are equivalent. It has been shown [3] that this is not the case and indeed a deep conceptual difference is present between mixing of states and mixing of fields. The results also extend to the mixing phenomenon of any particle, and are not limited to the case of Dirac neutrinos. Let us now consider the expansion for the Dirac fields ν_1 and ν_2 with definite masses appearing in Eqs.(3),(4):

$$\nu_i(x) = \sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\left[u^r_{\mathbf{k},i}(t)\alpha^r_{\mathbf{k},i} + v^r_{-\mathbf{k},i}(t)\beta^{r\dagger}_{-\mathbf{k},i}\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \quad i=1,2, \tag{6}$$

where $u^r_{\mathbf{k},i}(t) = e^{-i\omega_{k,i}t}u^r_{\mathbf{k},i}$ and $v^r_{-\mathbf{k},i}(t) = e^{i\omega_{k,i}t}v^r_{-\mathbf{k},i}$, with $\omega_{k,i} = \sqrt{\mathbf{k}^2+m_i^2}$. The $\alpha^r_{\mathbf{k},i}$ and the $\beta^r_{-\mathbf{k},i}$ $(r=1,2)$, are the annihilation operators for the vacuum state $|0\rangle_{1,2} \equiv |0\rangle_1\otimes|0\rangle_2$. Observe that Eqs.(1),(2) can be seen as arising by the application to the vacuum state $|0\rangle_{1,2}$ of the rotated operators:

$$R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},1}R(\theta) = \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1} + e^{-i\psi_k}\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},2}, \tag{7}$$
$$R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},2}R(\theta) = \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},2} - e^{i\psi_k}\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}, \tag{8}$$

and similar ones for $\beta^{r\dagger}_{\mathbf{k},i}$. An arbitrary phase ψ_k has been also included. The generator $R(\theta)$ is indeed the one of a rotation:

$$R(\theta) = \exp\left\{\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\}, \tag{9}$$

Notice that the unitary operator leaves the vacuum invariant : $R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}$. We now consider the action of the rotation Eq.(9) on the fields ν_1 :

$$R^{-1}(\theta)\nu_1(x)R(\theta) = \cos\theta\,\nu_1(x) + \sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}}\left(e^{i\psi_k}\alpha^r_{\mathbf{k},2}u^r_{\mathbf{k},1}(t) + e^{-i\psi_k}\beta^{r\dagger}_{\mathbf{k},2}v^r_{-\mathbf{k},1}(t)\right), \tag{10}$$

The above expressions do not fully reproduce the mixing at level of fields, cf.Eqs.(3): the problem is that the last term in the r.h.s. of Eq. (10) appears as the expansion of the field in the "wrong" basis. However, it is possible to recover the wanted expression by means of a suitable Bogoliubov transformation, which implements a mass shift. The generator(s) of which being

$$B_i(\Theta_i) = \exp\left\{\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\Theta_{\mathbf{k},i}\,\epsilon^r\left[\alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\} \quad i=1,2. \tag{11}$$

In fact,

$$\begin{aligned}
&B_2^{-1}(\Theta_2)\,R^{-1}(\theta)\,\nu_1(x)\,R(\theta)\,B_2(\Theta_2) = \\
&= \cos\theta\,\nu_1(x) + \sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}}\left(e^{i\psi_k}\tilde{\alpha}^r_{\mathbf{k},2}u^r_{\mathbf{k},1}(t) + e^{-i\psi_k}\tilde{\beta}^{r\dagger}_{\mathbf{k},2}v^r_{-\mathbf{k},1}(t)\right) \\
&= \cos\theta\,\nu_1(x) + \sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{k}\cdot\mathbf{x}}\left(e^{i\psi_k}\alpha^r_{\mathbf{k},2}\hat{u}^r_{\mathbf{k},1}(t) + e^{-i\psi_k}\beta^{r\dagger}_{\mathbf{k},2}\hat{v}^r_{-\mathbf{k},1}(t)\right)\,,
\end{aligned} \tag{12}$$

where

$$\hat{u}^r_{\mathbf{k},1}(t) = u^r_{\mathbf{k},1}e^{-i\omega_{k,1}t}e^{i\psi_k}\cos\Theta_{\mathbf{k},2} + \epsilon^r v^r_{-\mathbf{k},1}e^{i\omega_{k,1}t}e^{-i\phi_{k,2}}e^{-i\psi_k}\sin\Theta_{\mathbf{k},2}\,, \tag{13}$$
$$\hat{v}^r_{-\mathbf{k},1}(t) = v^r_{-\mathbf{k},1}e^{i\omega_{k,1}t}e^{-i\psi_k}\cos\Theta_{\mathbf{k},2} - \epsilon^r u^r_{\mathbf{k},1}e^{-i\omega_{k,1}t}e^{-i\phi_{k,2}}e^{-i\psi_k}\sin\Theta_{\mathbf{k},2}\,. \tag{14}$$

For $\hat{\Theta}_{\mathbf{k},2} = \cos^{-1}\left(e^{-i\psi_k}U_{\mathbf{k}}(t)\right)$, with $U_{\mathbf{k}}(t) \equiv u^{r\dagger}_{\mathbf{k},2}(t)u^r_{\mathbf{k},1}(t)$, the Bogoliubov transformation $B_2(\hat{\Theta}_2)$ produces the mass shift $m_2 - m_1$, such that$\hat{u}^r_{\mathbf{k},1}(t) = u^r_{\mathbf{k},2}(t)$ and $\hat{v}^r_{-\mathbf{k},1}(t) = v^r_{-\mathbf{k},2}(t)$. In definitive, the action of $B_2^{-1}(\hat{\Theta}_2)\,R^{-1}(\theta)$ produces the desired transformation of the field ν_1, cf. Eq.(3)[2]. This result is incomplete in that two different generators are needed for ν_1 and ν_2, whereas we know the algebraic generator for fields to be that of Eq.(5). It thus arises the problem of the decomposition of such generator in terms of rotation and Bogoliubov transformations: the full decomposition of the mixing generator is given by[4]

$$G(t;\theta,m_1,m_2) = B^{-1}(t;m_1,m_2)\,R(t;\theta)\,B(t;m_1,m_2)\,, \tag{15}$$

where the notation is now $f(\Theta_i(m_i)) \equiv f(m_i)$; $R(t;\theta)$ and $B(t;m_1,m_2)$ are defined as in Eqs.(9),(11), with the phases $\phi_{k,i} \equiv 2\omega_{k,i}t$ and $\psi_k \equiv (\omega_{k,1}-\omega_{k,2})t$ and the condition $\Theta_{\mathbf{k},i} = \frac{1}{2}\cot^{-1}(\frac{|\mathbf{k}|}{m_i})$ has been used [4]. From Eq.(15) it appears evident that the difference between G and R relies in the non zero value of the commutator $[R,B]$.

Vacuum structure and thermodynamical properties

It is possible to rewrite $G(\theta)$ (at $t=0$) as

$$G(\theta) = \exp\left\{2\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\left[U_{\mathbf{k}}J^r_{\mathbf{k},3} - \epsilon^r V_{\mathbf{k}}J^r_{\mathbf{k},2}\right]\right\}, \tag{16}$$

where we have introduced the following operators[3]:

$$J^r_{\mathbf{k},1} \equiv \frac{1}{2}\left[(\alpha^r_{\mathbf{k},1}\beta^r_{-\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},1}) - (\alpha^r_{\mathbf{k},2}\beta^r_{-\mathbf{k},2} - \beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2})\right], \tag{17}$$
$$J^r_{\mathbf{k},2} \equiv -\frac{1}{2}\left[(\alpha^r_{\mathbf{k},1}\beta^r_{-\mathbf{k},2} - \beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},1}) + (\alpha^r_{\mathbf{k},2}\beta^r_{-\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2})\right], \tag{18}$$
$$J^r_{\mathbf{k},3} \equiv \frac{1}{2}\left[(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2}) - (\alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1})\right], \tag{19}$$

which close the $su(2)$ algebra: $[J^r_{\mathbf{k},i}, J^r_{\mathbf{k},j}] = \epsilon_{ijk}J^r_{\mathbf{k},k}$ with $i,j,k = 1,2,3$. Moreover, considering that the Bogoliubov coefficients $U_{\mathbf{k}}$ and $V_{\mathbf{k}}$ appearing in Eq.(16) can be written as $U_{\mathbf{k}} = \cos(\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1})$, $V_{\mathbf{k}} = \sin(\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1})$, in the limit of small $(\Theta_{\mathbf{k},2}-\Theta_{\mathbf{k},1})$, it is possible to expand $V_{\mathbf{k}}$ in terms of the adimensional parameter $a \equiv \frac{(m_2-m_1)^2}{m_1m_2}$ so that $U_{\mathbf{k}} \cong 1$, $V_{\mathbf{k}} \cong a\tilde{V}_{\mathbf{k}}$, up to $o[(a)^2]$ where $\tilde{V}_{\mathbf{k}} \equiv \frac{|\mathbf{k}|\sqrt{m_1m_2}}{2(|\mathbf{k}|^2+m_1m_2)}$. Thus, looking at the tilde vacuum, defined as $|\tilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1,\Theta_2)|0\rangle_{1,2}$ where $B^{-1}(\Theta_1,\Theta_2) \equiv B_1^{-1}(\Theta_1)B_2^{-1}(\Theta_2)$:

$$|\tilde{0}\rangle_{1,2} \cong \left[1\!\!\mathrm{I} + \int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\sum_r\left(\Theta_{\mathbf{k},1}\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} + \Theta_{\mathbf{k},2}\,\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\right)\right]|0\rangle_{1,2}\,, \tag{20}$$

for $\Theta_{\mathbf{k},i}$ small, and comparing it with the flavor vacuum $|0\rangle_{e,\mu} \equiv G^{-1}(\theta)|0\rangle_{1,2}$ obtained in our approximation:

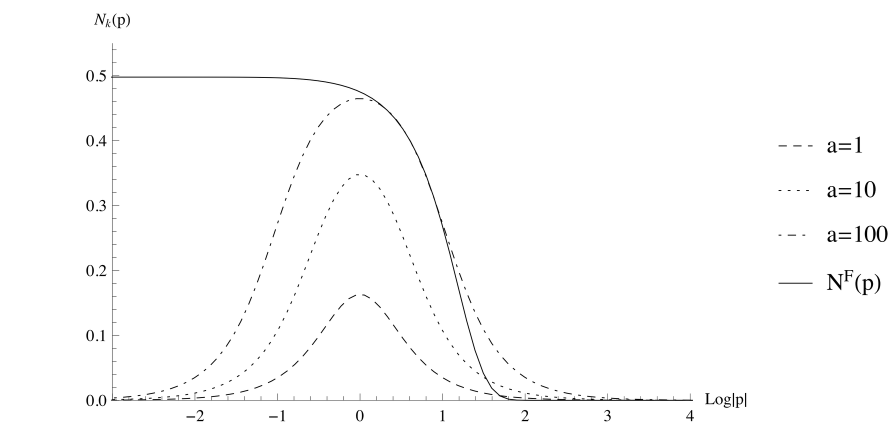
$$|0\rangle_{e,\mu} \cong \left[1\!\!\mathrm{I} + \theta\,a\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\,\tilde{V}_{\mathbf{k}}\sum_r\epsilon^r\left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)\right]|0\rangle_{1,2}\,. \tag{21}$$

we notice that, although the operatorial structure of the two above equations is similar, Eq.(21) exhibits non diagonal operatorial terms. From Eq.(21) we see that $|0\rangle_{e,\mu}$ cannot be reduced as a tensor product of vectors built on $|0\rangle_{1,2}$: this indeed confirms that the phenomenon of flavor mixing is related to the entanglement of mass eigenstates (see [5] for the discussion of entanglement in the context of particle mixing and oscillations). Another interesting feature of this phenomenon appears as one analyses more closely the parameter a, which is linked with the commutator $J^r_{\mathbf{k},2} = [J^r_{\mathbf{k},3}, J^r_{\mathbf{k},1}]$ which can be interpreted as a *non-diagonal Bogoliubov transformation*, and is the first non trivial term which contributes to the flavor vacuum structure. In order for a to exist it needs at least two fermion families to be present. In fact, with just one family the only adimensional parameter one can form is $|\mathbf{k}|/m$, which however depends on k and thus cannot be extracted from the integrals.

Now we examine the possibility of a thermodynamical interpretation for the condensate structure of the flavor vacuum, investigating the relation between the flavor vacuum and a thermal vacuum state of the form $|0(\beta_1,\beta_2)\rangle \equiv |0(\beta_1)\rangle\otimes|0(\beta_2)\rangle$ with

$$|0(\beta_i)\rangle \equiv \prod_{\mathbf{k},r}\left[\cos\gamma_{\mathbf{k},i}(\beta_i) + \sin\gamma_{\mathbf{k},i}(\beta_i)\,\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\right]|0\rangle_i\,, \tag{22}$$

where $i=1,2$ and $\gamma_{\mathbf{k},i}(\beta_i)$ are the parameters of the Bogoliubov transformations depending on the temperature. We recall [3] that it is possible to rewrite $|U_{\mathbf{k}}|^2$ in terms of two adimensional parameters: $|U_{\mathbf{k}}|^2 = \left(1 + 1/\sqrt{1+a(p/(p^2+1))^2}\right)/2$, with $p \equiv \frac{|\mathbf{k}|}{\sqrt{m_1m_2}}$, $a \equiv \frac{(m_2-m_1)^2}{m_1m_2}$. We consider the total number operator on the flavor vacuum $N_f(k) \equiv {}_{e,\mu}\langle 0|N_{\mathbf{k},1}+N_{\mathbf{k},2}|0\rangle_{e,\mu} = 2\sin^2\theta\,|V_{\mathbf{k}}|^2$ while the vev on the thermal vacuum gives[4] $N_F(k) \equiv \langle N_{\mathbf{k},1}+N_{\mathbf{k},2}\rangle_{\beta_1,\beta_2} = (e^{\beta_1\omega_{k,1}}+1)^{-1} + (e^{\beta_2\omega_{k,2}}+1)^{-1}$. One may wonder to what extent, $N_F(k)$ can fit $N_f(k)$ for given values of the parameters m_1, m_2 and θ, by adjusting the free parameters β_1 and β_2. From Fig. we see that this is somehow possible only for the right tail of the distribution $N_f(k)$; on the other hand, for low momenta, the behavior of the two distributions is quite different.



Plot of $N_f(p)$ and $N_F(p)$ against $Log|p|$. For all curves, we set $\theta = \pi/4$ and $m_1 = 20$. $N_f(p)$ is plotted for different values of a. For $N_F(p)$ we set $a = 100$, $T_1 = 10^4$ and $T_2 = 7.8\,10^4$.

This fact boils down to a structural difference between the two states $|0\rangle_{e,\mu}$ and $|0(\beta_1,\beta_2)\rangle$. These states differ because in the condensate structure of the "thermal" state $|0(\beta_1,\beta_2)\rangle$ are missing terms of the form $(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})|0\rangle_{1,2}$ due to the non-diagonal Bogoliubov transformation .

Conclusions and Outlook

We have discussed the algebraic structure of the mixing generator for two Dirac neutrino fields with different masses. It is interesting to observe that the Bogoliubov transformations are indeed responsible for the mass shift and thus the results of this paper can lead to further insight in the interplay between mixing phenomenon and mass generation in a dynamical perspective as recently discussed in Refs.[6]. Moreover, the condensate structure of the vacuum suggests a thermodynamical interpretation which we investigated, showing peculiarities in the thermal behavior due to the character of the particle-antiparticle condensate involved in the flavor vacuum. Such an issue will be further investigated in a future work along with the extension to the case of three flavors.

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[1] Our analysis is limited to the case of two Dirac neutrinos. Extension to three neutrinos is in our plans. However, we have good reasons to believe that the present results are general, since our arguments are of algebraic nature.

[2] A similar reasoning can be done from the start for ν_2, using $B_1^{-1}(\hat{\Theta}_1)\,R^{-1}(\theta)$, with $\hat{\Theta}_{\mathbf{k},1} = \cos^{-1}\left(e^{i\psi_k}U_{\mathbf{k}}(t)\right)$.

[3] We also have $J^r_{\mathbf{k},1} \equiv \frac{1}{2}(K^r_{\mathbf{k},1} - K^r_{\mathbf{k},2})$ with $K^r_{\mathbf{k},i} \equiv \alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}$ and $\ln B_i(\Theta_{\mathbf{k},i}) = \int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\Theta_{\mathbf{k},i}\sum_r K^r_{\mathbf{k},i}$; $\ln R(\theta) = 2\theta\int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\sum_r J^r_{\mathbf{k},3}$.

[4] F stands for Fermi. We use the notation $\langle 0(\beta_1,\beta_2)|*|0(\beta_1,\beta_2)\rangle \equiv \langle *\rangle_{\beta_1,\beta_2}$.