Renormalization Group Analysis of the Stochastic Navier-Stokes Equation with Colored Noise

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Fully developed turbrlence (1)

The turbulence is characterized by

- Cascades of energy;
- Scaling behaviour with universal "anomalous exponents";
- Intermittency.

The key parameters are

- W and L – power of the external source of energy and integral (external) scale; in troposphere $L \sim 1$ km;
- ν and l – viscosity coefficient and dissipation (internal) scale; in troposphere $l \sim 1$ cm.

Fully developed turbulence: $\mathrm{Re} \gg 1 \Rightarrow L \gg l \Rightarrow$ the inertial range $l \ll r \ll L$ exists.

Definition of the model: Stochastic equation (3)

The stochastic Navier–Stokes equation with random external force has the form

$$\partial_t v_i + (v_j \partial_j) v_i + \partial_i \wp = \nu_0 \partial^2 v_i + \phi_i,$$

where $v_i(x)$ is a transverse velocity field, ν_0 is molecular kinematic viscosity, $\wp$ is pressure, and ϕ_i is random foreign force (supplied the energy W in our system), with zero mean and preassigned correlator.

In Fourier space correlations of the random force ϕ_i can be represented in the form

$$\langle \phi_i(\omega, \mathbf{k})\, \phi_j(\omega', \mathbf{k}') \rangle \propto \delta(\omega + \omega')\delta(\mathbf{k} + \mathbf{k}') P_{ij}(\mathbf{k}) D_\phi(\omega, k),$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector.

Zero correlation time model (the simplest one):

$$D_\phi(\omega, k) = D_0 \frac{1}{k^{d+y}} \sim \delta(t - t').$$

Model with finite correlation time (colored noise):

$$D_\phi(\omega, k) = D_0 \frac{k^{8-d-y-2\eta}}{\omega^2 + \nu_0^2 u_0^2 k^{4-2\eta}};$$

here y and η are two UV regularizers, a new parameter u_0 is needed for the reason of dimensionality, and $D_0 > 0$ is an amplitude factor.

RG functions (6)

From the analysis of diagrams it follows that:

$$\beta_g = g(-y - 2\gamma_1 + 3\gamma_2); \quad \beta_u = u(-\eta + \gamma_2),$$

where

$$\gamma_1 = g \frac{u}{(u+1)^3}; \quad \gamma_2 = g \frac{u^3 d(d-1) + 3u^2 d(d-1) + 2u(d^2 - d + 2)}{4(u+1)^3}.$$

Here d is a dimension of $\mathbf{x}$ space.

Fixed points: saddle type fixed point (7)

From the analysis of β functions it follows that if $\frac{1}{3} < \alpha < \frac{1}{3} + \frac{4/3}{3d(d-1)+2}$ the system possesses fixed point $\{g^*, u^*\}$ with coordinates

$$u^* = \frac{-3 + \sqrt{1 - \frac{16(\alpha-1)}{d(d-1)(3\alpha-1)}}}{2}; \quad g^* = \frac{3\alpha - 1}{2} y \frac{(u^*+1)^3}{u^*},$$

where $\alpha = \eta / y$.

But one of the two eigenvalues of the matrix $\Omega = \partial \beta_i / \partial g_k$ for this point is less than zero, thus, this fixed point is saddle one, i.e., it is IR attractive only in one of the two possible directions.

The critical dimensions in this point are

$$\Delta_v = 1 + \frac{\eta - y}{2}, \quad \Delta_{v'} = d - 1 + \frac{\eta - y}{2}.$$

Kolmogorov–Obukhov theory (2)

The equal-time structure functions

$$S_n(\mathbf{r}) = \left\langle \left[v_r(t, \mathbf{x}) - v_r(t, \mathbf{x}') \right]^n \right\rangle,$$

where v_r is the component of the velocity field along the direction $\mathbf{r} = \mathbf{x} - \mathbf{x}'$.

From the two Kolmogorov's hypotesis (independence of L for $L \gg r$ and independence of l for $l \ll r$) it follows, that in the inertial range $l \ll r \ll L$

$$S_n(\mathbf{r}) = C_n\, (Wr)^{n/3}$$

with exact exponents and universal amplitudes C_n.

Due to the imtermittency statistical properties of the velocity are dominated by rare spatiotemporal configurations – the main contributions are given by infrequent, but strong events. This phenomenon is related with the strong fluctuations of the energy flux and leads to the violation of the classical K41 theory:

$$S_n(\mathbf{r}) = (Wr)^{n/3} (r/L)^{\gamma_n}$$

with [may be] singular dependence of L and an infinite set of "anomalous exponents" γ_n.

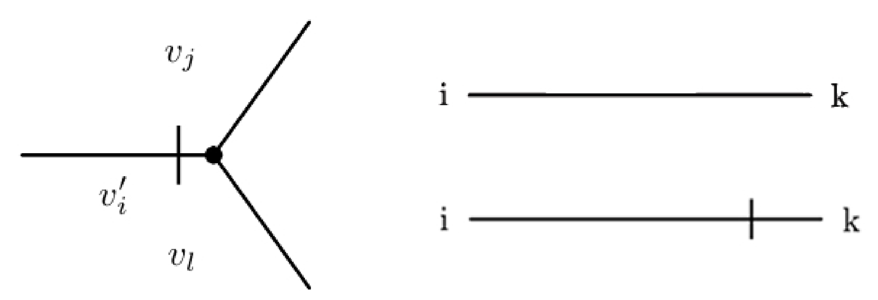
The goal is to calculate γ_n within a regular expansion.

Field theoretic formulation (4)

This stochastic problem is equivalent to the field theoretic model of the set of two fields $\Phi \equiv \{\boldsymbol{v}, \boldsymbol{v}'\}$ with action functional

$$\mathcal{S}(\Phi) = \frac{1}{2} v_i' D_v v_k' + v_k' \left[-\partial_t - (v_i \partial_i) + \nu_0 \partial^2 \right] v_k.$$

This model corresponds to a standard Feynman diagrammatic technique with the triple vertex and two bare propagators: $\langle v_i v_k' \rangle_0$ and $\langle v_i v_k \rangle_0$.

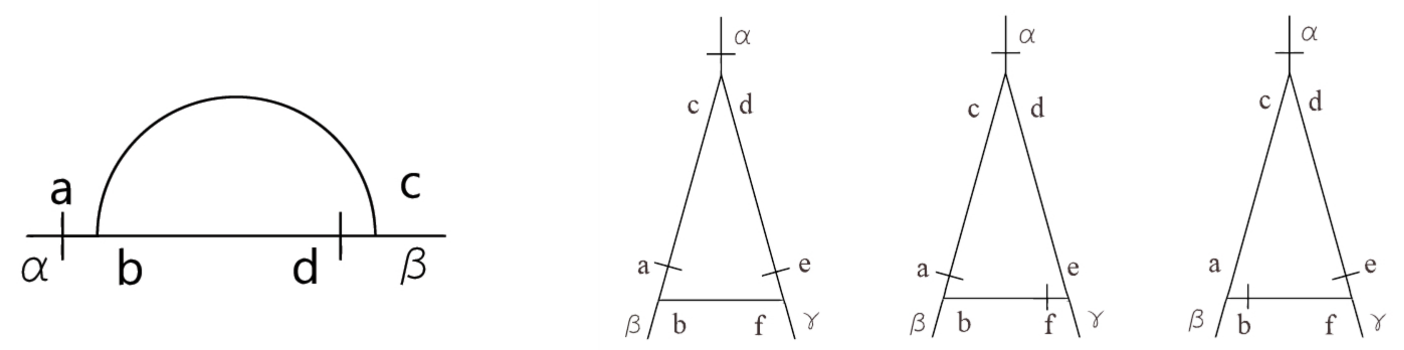


Divergences and diagrams (5)

For any $d > 2$ superficial divergences can be present only in the 1-irreducible functions of two types:

- $\langle v_\alpha' v_\beta \rangle_{1-\mathrm{ir}}$, for which the formal index of divergence is $d_\Gamma = 2$;
- $\langle v_\alpha' v_\beta v_\gamma \rangle_{1-\mathrm{ir}}$ with $d_\Gamma = 1$.

The one-loop approximation of these functions has the form



Fixed points: zero-time correlation fixed point (8)

Another case to be considered is $u^* \to \infty$. This case corresponds to the rapid-change model; this means that in this situation one should obtain the well-known fixed point of the model with zero-time correlations. This is indeed so:

$$\gamma_2 = g \frac{d(d-1)}{4} \quad \text{at} \quad u^* \to \infty, \quad \text{consequently} \quad g^* = y \frac{4(d+2)}{3(d-1)}.$$

This fixed point is IR attractive if $y > 0$, $\eta > y/3$.

In the leading order of IR asymptotic behavior, Green's functions satisfy the RG equation with the substitution $g \to g^*$ and $u \to u^*$. This feature, together with canonical scale invariance, gives us critical dimensions of the fields in the model, which, in fact, govern the asymptotic behaviour of arbitrary correlation functions. If $u^* \to \infty$ one obtains

$$\Delta_v = 1 - y/3, \quad \Delta_{v'} = d - 1 + y/3,$$

which is in agreement with well-known result for zero-time correlation model.

Symmetries of the Navier-Stokes equation (9)

This means, that the Galilean symmetry, destroyed by realistic random force with finite correlation time, restores in asymptotic behaviour – the only IR-attractive fixed point is a fixed point for zero-time correlation force. This result is in agreement with general idea that all symmetries of the Navier-Stokes equation, destroyed by boders, obstacles, and spontaneously, restores in the inertial range in statistical sense, i.e., for correlation and structure functions. See *U. Frisch, Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge Univ. Pr., Cambridge, 1995).