On the rôle of rotations and Bogoliubov transformations in neutrino mixing

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Outline[1]

- Pontecorvo vs QFT
- Rotation and Bogoliubov transformation
- Vacuum Structure and non commutativity
- Thermodynamical properties
- Conclusions

[1] Our analysis is limited to the case of two Dirac neutrinos. Extension to three neutrinos is in our plans. However, We believe that the present results are general, since our arguments are of algebraic nature.

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Pontecorvo vs QFT

Pontecorvo → mixing of states [2]

$$
\begin{aligned}
|\nu_e\rangle &= \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle, \\
|\nu_\mu\rangle &= \cos\theta\,|\nu_2\rangle - \sin\theta\,|\nu_1\rangle.
\end{aligned} \tag{1}
$$

Standard Model → mixing of fields [3]

$$
\begin{aligned}
\nu_e(x) &= \cos\theta\;\nu_1(x) + \sin\theta\;\nu_2(x)\,, \\
\nu_\mu(x) &= \cos\theta\;\nu_2(x) - \sin\theta\;\nu_1(x)\,,
\end{aligned} \tag{2}
$$

where $x \equiv (\mathbf{x}, t)$.

[2] M. S. Bilenky and B. Pontecorvo, Phys. Rept. (1978) **41** 225

[3] T. Cheng and L. Li,Gauge Theory of Elementary Particles Physics, Clarendon, Oxford, 1989.

Where ν_i are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} \left[u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \;+\; v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \right], \quad i = 1, 2.$$

Anticommutation, orthonormality and completeness relations are the standard ones.

To what extent the transformations in Eqs.(1) and (2) are equivalent?

Rotation

Eqs.(1) can be seen as arising by the application to the vacuum state $|0\rangle_{1,2}$ of the rotated operators:

$$R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},1}R(\theta) = \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},1} + e^{-i\psi_k}\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},2},$$
$$R(\theta)^{-1}\alpha^{r\dagger}_{\mathbf{k},2}R(\theta) = \cos\theta\,\alpha^{r\dagger}_{\mathbf{k},2} - e^{i\psi_k}\sin\theta\,\alpha^{r\dagger}_{\mathbf{k},1}, \quad (3)$$

and similar ones for $\beta^{r\dagger}_{\mathbf{k},i}$. An arbitrary phase ψ_k has been also included. The generator $R(\theta)$ is indeed the one of a rotation[4]:

$$R(\theta) = \exp\left\{\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2}+\beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2}\right)e^{i\psi_k}\right.\right.$$
$$\left.\left. - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1}+\beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\}, \quad (4)$$

[4] Notice that the unitary operator leaves the vacuum invariant: $R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}$

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[4] Notice that the unitary operator leaves the vacuum invariant: $R^{-1}(\theta)|0\rangle_{1,2} = |0\rangle_{1,2}$

Field rotation

The action of the rotation Eq.(4) on the fields ν_1

$$R^{-1}(\theta)\nu_1(x)R(\theta) = \cos\theta\,\nu_1(x) + \sin\theta\sum_r\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\left(e^{i\psi_k}\,\alpha^r_{\mathbf{k},2}\,u^r_{\mathbf{k},1}(t) + e^{-i\psi_k}\,\beta^{r\dagger}_{\mathbf{k},2}\,v^r_{-\mathbf{k},1}(t)\right), \qquad (5)$$

do not fully reproduce the mixing at level of fields, cf.Eqs.(2): the problem is that the last term in the r.h.s. of these equations appears as the expansion of the field in the “wrong” basis.

Bogoliubov transformation

However, it is possible to recover the wanted expression by means of a suitable Bogoliubov transformation, which implements a mass shift.

$$\begin{aligned} \tilde{\alpha}^{r\dagger}_{\mathbf{k},i} &= \cos\Theta_{\mathbf{k},i}\, \alpha^{r\dagger}_{\mathbf{k},i} - \epsilon^r\, e^{i\phi_{k,i}}\, \sin\Theta_{\mathbf{k},i}\, \beta^r_{-\mathbf{k},i}, \\ \tilde{\beta}^{r\dagger}_{-\mathbf{k},i} &= \cos\Theta_{\mathbf{k},i}\, \beta^{r\dagger}_{-\mathbf{k},i} + \epsilon^r\, e^{-i\phi_{k,i}}\, \sin\Theta_{\mathbf{k},i}\, \alpha^r_{\mathbf{k},i}, \end{aligned} \tag{6}$$

with $\tilde{\alpha}^{r\dagger}_{\mathbf{k},i} \equiv B_i^{-1}(\Theta_i)\, \alpha^{r\dagger}_{\mathbf{k},i}\, B_i(\Theta_i)$, $\tilde{\beta}^{r\dagger}_{-\mathbf{k},i} \equiv B_i^{-1}(\Theta_i)\beta^{r\dagger}_{-\mathbf{k},i}\, B_i(\Theta_i)$, $i = 1, 2$ and the generator(s)

$$B_i(\Theta_i) = \exp\left\{ \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \Theta_{\mathbf{k},i}\, \epsilon^r \left[\alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i} e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i} e^{i\phi_{k,i}} \right] \right\}.$$

Rotation and Bogoliubov transformation

In fact,

$$B_2^{-1}(\Theta_2)\, R^{-1}(\theta)\, \nu_1(x)\, R(\theta)\, B_2(\Theta_2) =$$

$$= \cos\theta\, \nu_1(x) + \sin\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(e^{i\psi_k}\, \tilde{\alpha}^r_{\mathbf{k},2} u^r_{\mathbf{k},1}(t) + e^{-i\psi_k}\, \tilde{\beta}^{r\dagger}_{\mathbf{k},2} v^r_{-\mathbf{k},1}(t) \right)$$

$$= \cos\theta\, \nu_1(x) + \sin\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}} \left(e^{i\psi_k}\, \alpha^r_{\mathbf{k},2} \hat{u}^r_{\mathbf{k},1}(t) + e^{-i\psi_k}\, \beta^{r\dagger}_{\mathbf{k},2} \hat{v}^r_{-\mathbf{k},1}(t) \right), \quad (7)$$

where

$$\hat{u}^r_{\mathbf{k},1}(t) = u^r_{\mathbf{k},1} e^{-i\omega_{k,1}t} e^{i\psi_k} \cos\Theta_{\mathbf{k},2} + \epsilon^r v^r_{-\mathbf{k},1} e^{i\omega_{k,1}t} e^{-i\phi_{k,2}} e^{-i\psi_k} \sin\Theta_{\mathbf{k},2}\,,$$

$$\hat{v}^r_{-\mathbf{k},1}(t) = v^r_{-\mathbf{k},1} e^{i\omega_{k,1}t} e^{-i\psi_k} \cos\Theta_{\mathbf{k},2} - \epsilon^r u^r_{\mathbf{k},1} e^{-i\omega_{k,1}t} e^{-i\phi_{k,2}} e^{-i\psi_k} \sin\Theta_{\mathbf{k},2}\,. \quad (8)$$

For

$$\hat{\Theta}_{\mathbf{k},2} = \cos^{-1}\left(e^{-i\psi_k} U_{\mathbf{k}}(t)\right)$$

with $U_{\mathbf{k}}(t) \equiv u^{r\dagger}_{\mathbf{k},2}(t) u^{r}_{\mathbf{k},1}(t)$, the Bogoliubov transformation $B_2(\hat{\Theta}_2)$ produces the mass shift

$$\Delta m = m_2 - m_1$$

such that

$$\hat{u}^{r}_{\mathbf{k},1}(t) = u^{r}_{\mathbf{k},2}(t) \;\text{ and }\; \hat{v}^{r}_{-\mathbf{k},1}(t) = v^{r}_{-\mathbf{k},2}(t).$$

In definitive, the action of $B_2^{-1}(\hat{\Theta}_2)\, R^{-1}(\theta)$ produces the desired transformation of the field ν_1, cf. Eq.(2).[5]

[5] A similar reasoning can be done for ν_2, using $B_1^{-1}(\hat{\Theta}_1)\, R^{-1}(\theta)$, with $\hat{\Theta}_{\mathbf{k},1} = \cos^{-1}\left(e^{i\psi_k} U_{\mathbf{k}}(t)\right)$.

Mixing generator

But two different generators are needed for ν_1 and ν_2, whereas we know the mixing generator for fields to be[6]:

$$G(t;\theta,m_1,m_2) = \exp\left\{\theta \int d^3\mathbf{x} \left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right\} .$$

whose full decomposition is given by [7] [8]

$$G(t;\theta,m_1,m_2) = B^{-1}(t;m_1,m_2)\, R(t;\theta)\, B(t;m_1,m_2)\,,$$

difference between G and $R \leftrightarrow [R,B] \neq 0$.

[6] M.Blasone and G.Vitiello, Annals Phys. (1995)

[7] M. Blasone, M.V. Gargiulo and G. Vitiello, J. Phys. Conf. Ser. (2015) **626** 012026.

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Where[9]:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r}\left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^{r}_{-\mathbf{k},2}\right)e^{i\psi_k} + -\left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^{r}_{-\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r}\Theta_{\mathbf{k},i}\;\epsilon^r\left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\},\quad i = 1,2$$

Since $[B_1, B_2] = 0$ we put $B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$

9

$$\psi_k = (\omega_{k,1} - \omega_{k,2})\, t); \qquad \phi_{k,i} = 2\,\omega_{k,i}\, t; \qquad \Theta_{\mathbf{k},i} = \frac{1}{2}\cot^{-1}\left(\frac{|\mathbf{k}|}{m_i}\right)$$

The $B_i(\Theta_{\mathbf{k},i})$, $i = 1, 2$ are ordinary Bogoliubov transformations which introduce a mass shift, and $R(\theta)$ is a rotation.

Their action on the vacuum is given by:

$$|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} = \prod_{\mathbf{k},r} \Big[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}\beta^{r\dagger}_{-\mathbf{k},i}\Big]|0\rangle_{1,2}$$

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R^{-1}(\theta)|0\rangle_{1,2} &= |0\rangle_{1,2} \, .
\end{aligned}
$$

The full expansion of the mixing generator $G(\theta)$ (at $t = 0$) is

$$\begin{aligned} G_\theta \;=\;& \exp\Big\{ \sum_r \Big(U_{\mathbf{k}}\; \alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} - \epsilon^r\; V_{\mathbf{k}}\; \beta^{r}_{-\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \epsilon^r\; V_{\mathbf{k}}\; \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + U_{\mathbf{k}}\; \beta^{r}_{-\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\Big) \\ -\;& \sum_r \Big(U_{\mathbf{k}}\; \alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \epsilon^r\; V_{\mathbf{k}}\; \beta^{r}_{-\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} - \epsilon^r\; V_{\mathbf{k}}\; \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} + U_{\mathbf{k}}\; \beta^{r}_{-\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big) \Big\} \end{aligned}$$

It is possible to rewrite it as

$$G(\theta) = \exp\left\{ 2\theta \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[U_{\mathbf{k}} J^r_{\mathbf{k},3} - \epsilon^r V_{\mathbf{k}} J^r_{\mathbf{k},2} \right] \right\},$$

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Where we have introduced the following operators

$$J^r_{\mathbf{k},1} \equiv \frac{1}{2}\Big[(\alpha^r_{\mathbf{k},1}\beta^r_{-\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},1}) - (\alpha^r_{\mathbf{k},2}\beta^r_{-\mathbf{k},2} - \beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2})\Big],$$

$$J^r_{\mathbf{k},2} \equiv -\frac{1}{2}\Big[(\alpha^r_{\mathbf{k},1}\beta^r_{-\mathbf{k},2} - \beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},1}) + (\alpha^r_{\mathbf{k},2}\beta^r_{-\mathbf{k},1} - \beta^{r\dagger}_{-\mathbf{k},1}\alpha^{r\dagger}_{\mathbf{k},2})\Big],$$

$$J^r_{\mathbf{k},3} \equiv \frac{1}{2}\Big[(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^r_{\mathbf{k},2} + \beta^{r\dagger}_{-\mathbf{k},1}\beta^r_{-\mathbf{k},2}) - (\alpha^{r\dagger}_{\mathbf{k},2}\alpha^r_{\mathbf{k},1} + \beta^{r\dagger}_{-\mathbf{k},2}\beta^r_{-\mathbf{k},1})\Big],$$

which close the $su(2)$ algebra: $[J^r_{\mathbf{k},i}, J^r_{\mathbf{k},j}] = \epsilon_{ijk}J^r_{\mathbf{k},k}$ with $i,j,k = 1,2,3$.
Notice that

$$\ln B_i(\Theta_{\mathbf{k},i}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\Theta_{\mathbf{k},i}\sum_r K^r_{\mathbf{k},i}\ , \quad \ln R(\theta) = 2\theta \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\sum_r J^r_{\mathbf{k},3}$$

with $K^r_{\mathbf{k},i} \equiv \alpha^r_{\mathbf{k},i}\beta^r_{-\mathbf{k},i} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}$ and $J^r_{\mathbf{k},1} \equiv \frac{1}{2}(K^r_{\mathbf{k},1} - K^r_{\mathbf{k},2})$

Moreover, considering that the Bogoliubov coefficients can be written as

$$U_{\mathbf{k}} = \cos(\Theta_{\mathbf{k},2} - \Theta_{\mathbf{k},1}) \, , \; V_{\mathbf{k}} = \sin(\Theta_{\mathbf{k},2} - \Theta_{\mathbf{k},1})$$

In the limit of small $(\Theta_{\mathbf{k},2} - \Theta_{\mathbf{k},1})$, it is possible to expand $V_{\mathbf{k}}$ in terms of the adimensional parameter

$$a \equiv \frac{(m_2 - m_1)^2}{m_1 m_2}$$

so that

$$U_{\mathbf{k}} \cong 1 \, , \; V_{\mathbf{k}} \cong a\tilde{V}_{\mathbf{k}}$$

up to $o[(a)^2]$ where $\tilde{V}_{\mathbf{k}} \equiv \frac{|\mathbf{k}|\sqrt{m_1 m_2}}{2(|\mathbf{k}|^2 + m_1 m_2)}$

Thus,

$$G(\theta) \cong \mathbb{1} + 2\theta \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sum_r J^r_{\mathbf{k},3} + 2\theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r J^r_{\mathbf{k},2}$$

θ and a : physical parameter of the transformation

$$\theta = 0 \rightarrow G(\theta) \cong \mathbb{1} \qquad a = \frac{(m_2 - m_1)^2}{m_1 m_2} = 0 \rightarrow G(\theta) \cong R(\theta)$$

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appears at second order in the expansion, being linked with the commutator

$$J^r_{\mathbf{k},2} = [J^r_{\mathbf{k},3}, J^r_{\mathbf{k},1}]$$

non-diagonal Bogoliubov transformation

first non trivial term which contributes to the flavor vacuum structure.

in order to exist needs at least two fermion families to be present.

One family $\leftrightarrow \frac{|\mathbf{k}|}{m}$

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One family $\leftrightarrow \frac{|\mathbf{k}|}{m}$

For $\Theta_{\mathbf{k},i}$ small.
The tilde vacuum, defined as $|\tilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2}$[10] is

$$|\tilde{0}\rangle_{1,2} \cong \left[\mathbf{I} + \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sum_r \left(\Theta_{\mathbf{k},1}\, \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} + \Theta_{\mathbf{k},2}\, \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\right)\right]|0\rangle_{1,2}\,,$$

The flavor vacuum $|0\rangle_{e,\mu} \equiv G^{-1}(\theta)|0\rangle_{1,2}$ is:

$$|0\rangle_{e,\mu} \cong \left[\mathbf{I} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\; \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r \left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)\right]|0\rangle_{1,2}\,.$$

$|0\rangle_{e,\mu}$ cannot be reduced as a tensor product of vectors built on $|0\rangle_{1,2}$
phenomenon of flavor mixing is related to the entanglement of mass eigenstates.

10 $B^{-1}(\Theta_1, \Theta_2) \equiv B_1^{-1}(\Theta_1)B_2^{-1}(\Theta_2)$

For $\Theta_{\mathbf{k},i}$ small.
The tilde vacuum, defined as $|\tilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2}$[10] is

$$|\tilde{0}\rangle_{1,2} \cong \left[\mathbf{I} + \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \sum_r \left(\Theta_{\mathbf{k},1}\, \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} + \Theta_{\mathbf{k},2}\, \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}\right)\right]|0\rangle_{1,2}\,,$$

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Thermodynamical properties

Now we investigate the possibility of a thermodynamical interpretation studying the relation between the flavor vacuum and a thermal vacuum state of the form

$$|0(\beta_1, \beta_2)\rangle \equiv |0(\beta_1)\rangle \otimes |0(\beta_2)\rangle$$

with

$$|0(\beta_i)\rangle \equiv \prod_{\mathbf{k},r} \Big[\cos\gamma_{\mathbf{k},i}(\beta_i) + \sin\gamma_{\mathbf{k},i}(\beta_i)\, \alpha^{r\dagger}_{\mathbf{k},i} \beta^{r\dagger}_{-\mathbf{k},i} \Big] |0\rangle_i \,,$$

where $i = 1, 2$ and $\gamma_{\mathbf{k},i}(\beta_i)$ are the parameters of the Bogoliubov transformations depending on the temperature.

The total number operator on the flavor vacuum is

$$N_f(k) \equiv {}_{e,\mu}\langle 0| N_{\mathbf{k},1} + N_{\mathbf{k},2} |0\rangle_{e,\mu} = 2\,\sin^2\theta\,|V_{\mathbf{k}}|^2$$

The vev on the thermal vacuum gives[11]

$$N_F(k) \equiv \langle N_{\mathbf{k},1} + N_{\mathbf{k},2}\rangle_{\beta_1,\beta_2} = \frac{1}{e^{\beta_1\omega_{k,1}}+1} + \frac{1}{e^{\beta_2\omega_{k,2}}+1}$$

One may wonder to what extent, $N_F(k)$ can fit $N_f(k)$ for given values of the parameters m_1, m_2 and θ, by adjusting the free parameters β_1 and β_2.

[11] F stands for Fermi. We use the notation $\langle 0(\beta_1,\beta_2)| * |0(\beta_1,\beta_2)\rangle \equiv \langle * \rangle_{\beta_1,\beta_2}$.

We see that this is somehow possible only for the right tail of the distribution $N_f(k)$; on the other hand, for low momenta, the behavior of the two distributions is quite different[12]

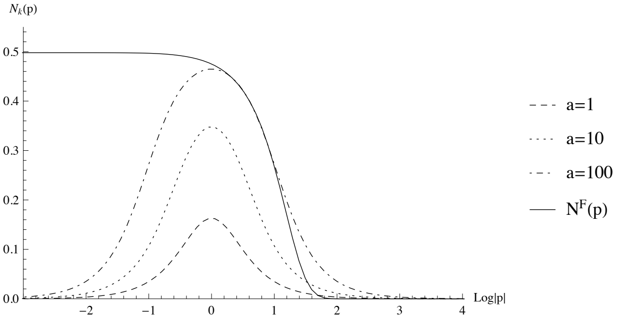

Figure: Plot of $N_f(p)$ and $N_F(p)$ against $Log|p|$. For all curves, we set $\theta = \pi/4$ and $m_1 = 20$. $N_f(p)$ is plotted for different values of a. For $N_F(p)$ we set $a = 100$, $T_1 = 10^4$ and $T_2 = 7.8\,10^4$.

[12] $|U_{\mathbf{k}}|^2 = \left(1 + 1/\sqrt{1 + a(p/(p^2+1))^2}\right)/2$; $p \equiv \frac{|\mathbf{k}|}{\sqrt{m_1 m_2}}$, $a \equiv \frac{(m_2 - m_1)^2}{m_1 m_2}$.

Structural difference between the two states $|0\rangle_{e,\mu}$ and $|0(\beta_1, \beta_2)\rangle$.

These states differ because in the condensate structure of the "thermal" state $|0(\beta_1, \beta_2)\rangle$ are missing terms of the form

$$(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})|0\rangle_{1,2}$$

due to the non-diagonal Bogoliubov transformation.

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Conclusion

- Incompatibility of the mixing transformation as mere rotations both for states and fields. ⇔ Necessity of implementing a mass shift.
- The explicit dependance on the true physical parameters of the mixing transformation, i.e. θ and a.
- a appears at second order in the expansion, being linked with the *non-diagonal Bogoliubov transformation*, and is the first non trivial term which contributes to the flavor vacuum structure.
- Peculiarities in the thermal behavior due to the character of the particle-antiparticle condensate involved in the flavor vacuum.

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- Further investigation on the thermodynamical properties
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Thank you
for your kind attention