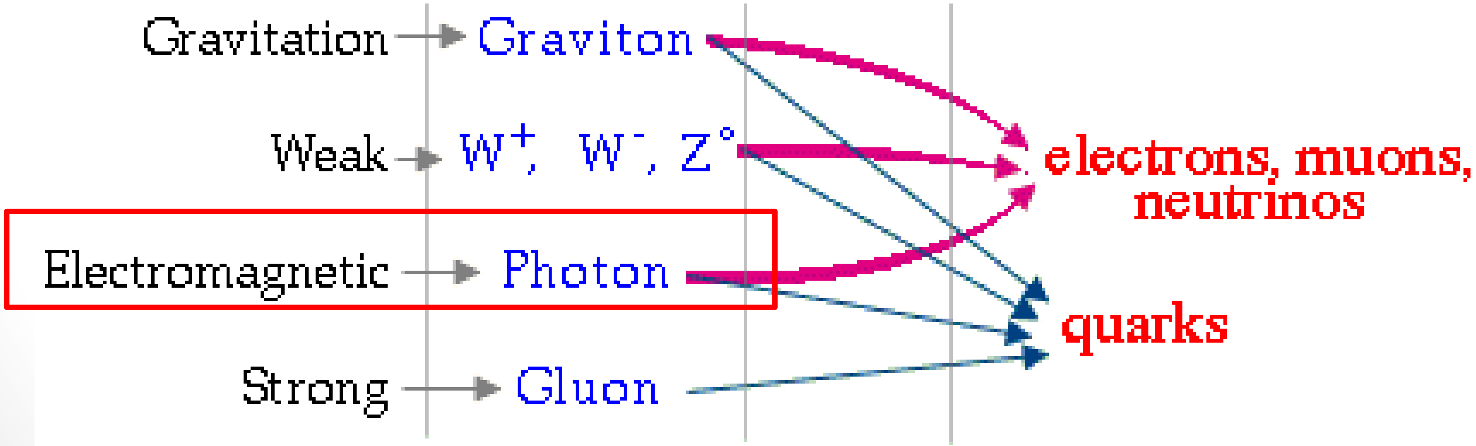
Finite Temperature Effects on Lepton Interactions

Maria Naeem
Department of Physics
COMSATS Institute of Information Technology, Islamabad, Pakistan

Types of Interactions

- Gravitational Interaction
- Strong Interaction
- **Electromagnetic Interaction**
- Weak Interaction



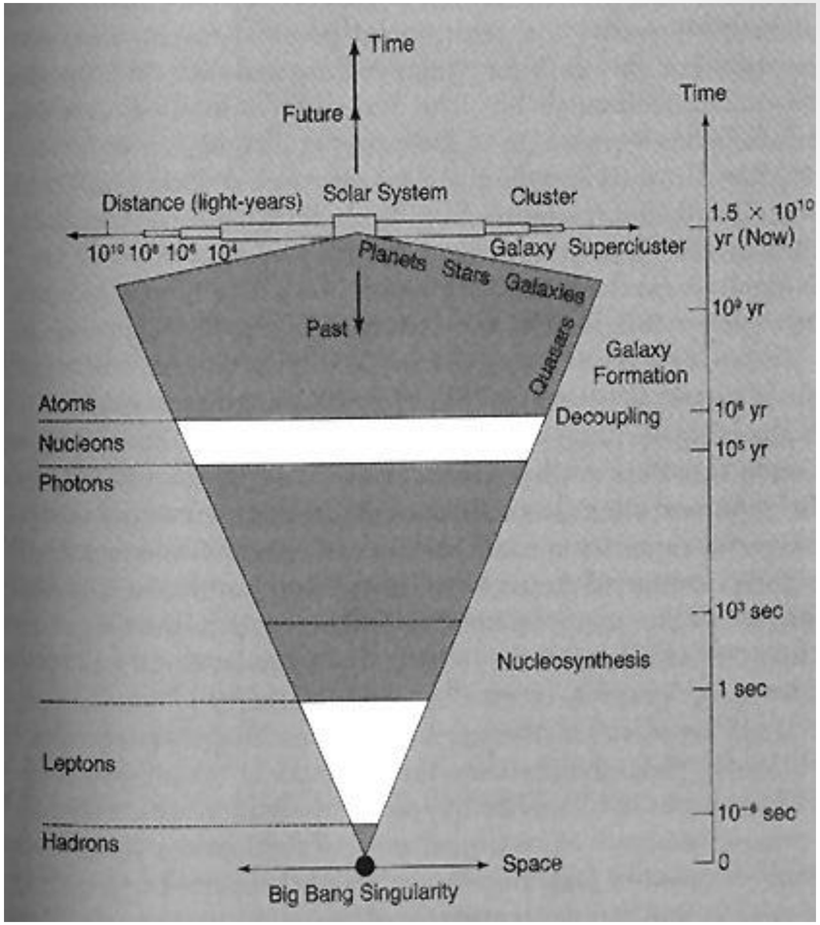
Motivation

- **Lepton epoch**
  - After the Big Bang

    $10^{-4} - 100$ sec

    $10^{10} - 10^{12}$ K

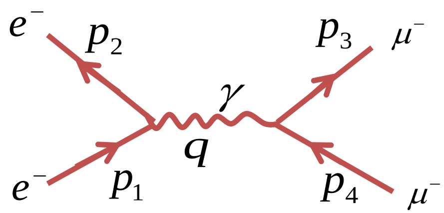
    $1 - 100$ MeV



- **Experimental setups**
  - SLAC Large Detector (SLD) at Stanford Linear Collider from 1992-1998 (91.2GeV).
  - Large Electron-Positron Collider (LEP) in CERN operated from 1989 – 2000: started at cm energy (91GeV), upgraded to (100 – 209GeV)
  - The Compact Linear Collider at CERN (CLIC) (3TeV)
  - International Linear Collider in (probably) Japan (ILC) (500GeV –1TeV)

Elastic Scattering on QED Processes

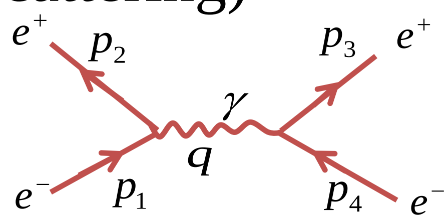
- Non relativistic scattering
  - Rutherford Scattering



- Elastic relativistic scatterings
  - Electron Muon Scattering (Mott Scattering)

$$e^- \mu^- \rightarrow e^- \mu^-$$

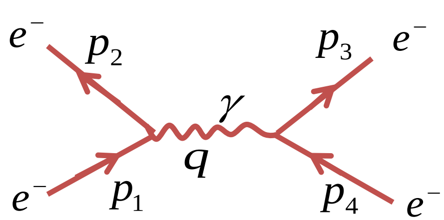
  - Electron positron scattering (Moller Scattering)



$$e^- e^+ \rightarrow e^- e^+$$

  - Electron electron Scattering (Bhabha Scattering)

$$e^- e^- \rightarrow e^- e^-$$

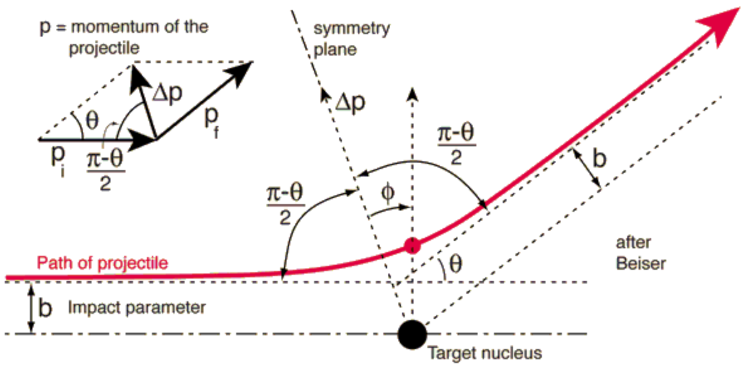


Elastic Scatterings in QED

Rutherford Scattering

- Consider the classical case of e^- encountering Coulomb potential of stationary p and scattered off at an angle θ

$$\sigma = \pi b^2 = \pi \left(\frac{Zze^2}{2E} \right)^2 \cot^2 \frac{\theta}{2}$$



- For calculating the differential cross-section one define

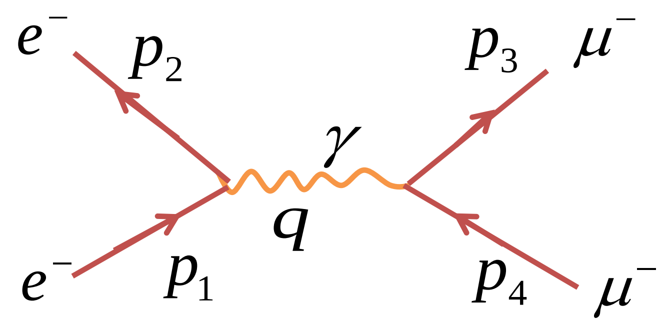
$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\theta}\frac{d\theta}{d\Omega} \quad \text{with} \quad d\Omega = \sin\theta d\theta d\phi$$

- Therefore we finally get

$$\frac{d\sigma}{d\Omega} = \left(\frac{Zze^2}{4E\sin^2\frac{\theta}{2}} \right)^2$$

Elastic Scattering on QED Processes

- Mott Scattering $e^- \mu^- \rightarrow e^- \mu^-$



$$M = \frac{-e^2}{(p_1 + p_2)^2}[\bar{u}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma^\mu u(p_4)]$$

$$|M|^2 = \frac{8e^2}{s^2}\left\{(p_1.p_3)(p_2.p_4) + (p_1.p_4)(p_2.p_3) - m^2(p_3.p_4) - m_\mu{}^2(p_1.p_2) + 2m^2 m_\mu{}^2\right\}$$

Simplifying in terms of Mandelstam variables

$$p_1.p_2 = \frac{s}{2} - m^2 \qquad p_3.p_4 = \frac{s}{2} - m_\mu{}^2 \qquad p_1.p_4 = p_2.p_3 = \frac{1}{2}\left(m^2 + m_\mu{}^2 - u\right)$$

$$p_1.p_3 = p_2.p_4 = \frac{1}{2}\left(m^2 + m_\mu{}^2 - t\right)$$

$$|M|^2 = \frac{2e^2}{s^2}\left\{t^2 + u^2 + 2s(m^2 + m_\mu{}^2) - 2(m^2 - 6m^2 m_\mu{}^2 + m_\mu{}^4)\right\}$$

Differential cross section formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{64\pi^2 E_{cm}{}^2}|M|^2 \frac{p_f}{p_i}$$

In cm we have two degrees of freedom: energy and angle θ, so that

$$p_1 = (E,\mathbf{k}) \qquad p_2 = (E,-\mathbf{k}) \qquad \text{where} \qquad |\mathbf{k}| = \sqrt{E^2 - m^2}$$

and

$$p_3 = (E,\mathbf{p}) \qquad p_4 = (E,-\mathbf{p}) \qquad \text{with} \qquad |\mathbf{p}| = \sqrt{E^2 - m_\mu{}^2}$$

then

$$s = 4E^2 = E_{cm}^2 \qquad t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = m^2 + m_\mu{}^2 - 2E^2 + 2\mathbf{p}.\mathbf{k}$$

Differential cross section becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\alpha^2}{2E^6}\left|\frac{\vec{p}}{\vec{k}}\right|\left[4\left\{E^4 + (\mathbf{p}.\mathbf{k})^2 + 2m^2 m_\mu{}^2\right\}\right] \qquad \text{with} \qquad e^2 = 4\pi\alpha$$

The angular dependence comes from the term $\mathbf{p}.\mathbf{k} = |\mathbf{p}||\mathbf{k}|\cos\theta$

Finally

$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2}{E^6}\frac{\sqrt{E^2 - m^2}}{\sqrt{E^2 - m_\mu{}^2}}\left[\left\{E^4 + (E^2 - m^2)(E^2 - m_\mu{}^2)\cos^2\theta + 2m^2 m_\mu{}^2\right\}\right]$$

- Integrating over both the sides the cross section becomes

$$\sigma = \frac{8\pi\alpha^2}{3E^6}\left[4E^4 + 7m^2 m_\mu{}^2 + E^2(3m_\mu{}^2 - m^2)\right]$$

- The result reduces to the case of Rutherford scattering for nonrelativistic limit if

$$m << m_\mu$$

Here

$$m = 0.51 MeV$$

while

$$m_\mu = 105.7 MeV$$

Quantum Field Theory

- Main framework relies on the particle creation and annihilation operators corresponding to the field operators:

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_k}} \left[a(k) e^{-ik.x} + a^\dagger(k) e^{ik.x} \right]$$

$$\Phi(y) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_k}} \left[a(k) e^{-ik.y} + a^\dagger(k) e^{ik.y} \right]$$

- Fourier transformation of field operator from space (*x*,*y*) to momentum (*k*)
- $a(k), a^\dagger(k)$ are the respective operators for spontaneous creation and annihilation of the elementary particles.
- Green function are introduced corresponding to a field propagator from *x* to *y* space-time $i\Delta_F(x-y) = \langle 0 | T\{\Phi(x)\Phi(y)\} | 0 \rangle$
- Putting values for the fields in terms of the creation and annihilation operators and using the commutation relations, we get finally

$$\Delta(x-y) = \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2 - m^2}$$

Finite Temperature QED

- Assumption of background of e^-, e^+ and $\gamma's$

- Non zero probability of virtual particles replacement with real particles in the medium

- Propagators are modified in Finite Temperature (FT) Field Theory to include statistical effects from the background

FT conditions get included in following ways:

- Free spinors get replaced by FT spinors
- FT propagators used in place of usual propagators at $T=0$
- FT modifies the cross-section through radiative corrections

Formalisms for FT

- The statistical effects are calculated either in

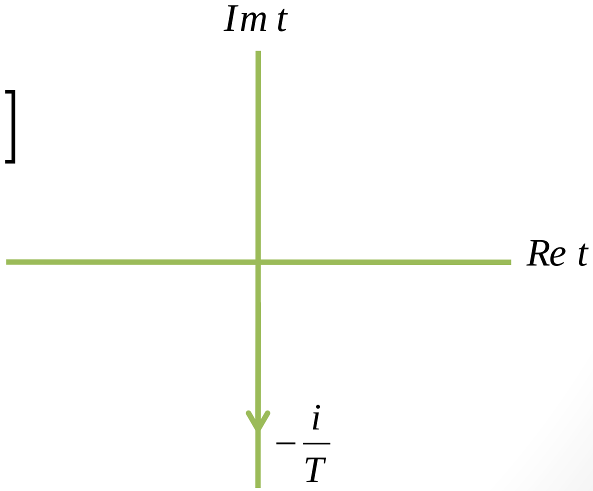
  Euclidean space or Minkowski space

using

  Imaginary or Real time formalisms, respectively.

- With the path integral quantization of field theories:
  - the statistical contribution in FT bath is included using partition function $Z = Tr(e^{-\beta H})$
- FT propagator

$$i\Delta_F^{T>0}(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{1}{\sqrt{2E_k}} \left[(1+n_B) e^{-ik.(x-y)} + n_B e^{ik.(x-y)} \right]$$



Bose-Einstein distribution function

$$n_B(E_k) = \frac{1}{e^{\beta E_k} - 1}$$

Real time formalism

- Energy is a continuous variable as in conventional field theory
- Contains temperature dependence as separate additional term with the vacuum term

Boson Propagator

$$\Delta(x-y)=\left[\frac{i}{k^2-m+i\varepsilon}+2\pi\, n_B(E_k)\delta\left(k^2-m^2\right)\right]$$

Fermion Propagator

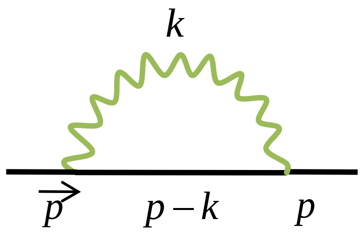
$$\Delta(x-y)=\int\frac{d^4k}{(2\pi)^4}(\not{p}+m)\left[\frac{i}{p^2-m+i\varepsilon}+2\pi\, n_F(E_p)\delta\left(p^2-m^2\right)\right]$$

- Fermi-Dirac distribution function

$$n_F(E_p)=\frac{1}{e^{\beta E_p}+1}$$

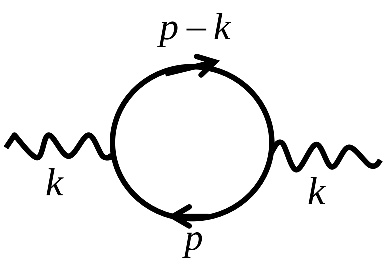
QED Radiative Corrections

- Electron Self Energy



$$\Sigma(p) = \frac{e^2}{(2\pi)^4}\int d^4k\, \gamma_\mu \frac{(\not{p} - \not{k} + m)}{\left((p-k)^2 - m^2\right)k^2}\gamma^\mu$$

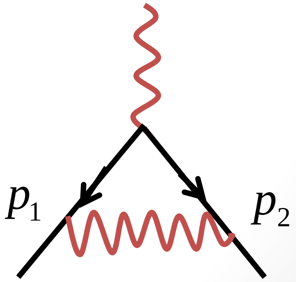
- Vacuum polarization



$$\Pi^{\mu\nu}(p) = \frac{ie^2}{(2\pi)^4}\int d^4k\, Tr\left(\frac{\gamma^\mu(\not{p} + \not{k} + m)\gamma_\nu(\not{p} + m)}{((p+k)^2 - m^2)(p^2 - m^2)}\right)$$

- Vertex correction

$$M^{\mu\nu} = \frac{ie^2}{(2\pi)^4}\int d^4k \left(\frac{\gamma_\beta(\not{p}_1 - \not{k} + m)\gamma^\mu(\not{p}_2 - \not{k} + m)\gamma_\alpha}{((p_1 - k)^2 - m^2)(\not{p}_2 - k)^2 - m^2)k^2)}\right)$$



QED Radiative Corrections at FT

Electron Self Energy

- Electron self-energy calculated at FT using real time formalism
- The temperature dependent part in self-energy can be written in a compact form as follows

$$\Sigma_\beta(p) = \Sigma_{T=0}(p) + \frac{e^2}{4\pi^3}\int d^4k\left(2m - \not p - \not k\right)\left[\frac{n_F(E_{p-k})\,\delta[(p-k)^2 - m^2]}{k^2 + i\varepsilon} - \frac{n_B(k)\,\delta(k^2)}{(p-k)^2 - m^2 + i\varepsilon}\right]$$

$$= \Sigma_{T=0}(p) + \frac{\alpha}{4\pi^2}\left[(\not p - m)I_A + \not I + (2m - \not p)\not J_A + \not J_B\right]$$

where

$$I^A = 8\pi\int\frac{dk}{k}n_B(k) \qquad I^\mu = 2\int\frac{d^3k}{k}n_B(k)\frac{\left(k_0, \vec{k}\right)}{E_p k_0 - p.k}$$

$$J_A = \int\frac{d^3l}{E_l}n_F(E_l)\left[\frac{1}{E_pE_l + m^2 - \vec{p}.\vec{l}} - \frac{1}{E_pE_l - m^2 + \vec{p}.\vec{l}}\right]$$

$$J_B = \int\frac{d^3l}{E_l}n_F(E_l)\left[\frac{E_p + E_l, \vec{p} + \vec{l}}{E_pE_l + m^2 - \vec{p}.\vec{l}} - \frac{E_p - E_l, \vec{p} + \vec{l}}{E_pE_l - m^2 + \vec{p}.\vec{l}}\right]$$

We can write the self energy as

$$\Sigma(p) = A(p)E\gamma_0 - B(p)\mathbf{p}.\boldsymbol{\gamma} - C(p)$$

with

$$A = \frac{\alpha}{4\pi^2}\left[I^A + \frac{1}{E}I^0 - J_A + \frac{J_B^0}{E}\right] \qquad B = \frac{\alpha}{4\pi^2}\left[I^A + \frac{1}{p^2}I.p - J_A + \frac{1}{p^2}J_B.p\right]$$

$$C = \frac{\alpha}{4\pi^2}m\left[I^A - 2J^A\right]$$

Now taking inverse of fermion propagator as

$$S^{-1}(p) - (1-A)E\gamma^0 - (1-B)\mathbf{p}.\boldsymbol{\gamma} - (m-C) = \not{p} - m$$

Momentum term can be separated from mass term:

$$p = ((1-A)E, (1-B)\mathbf{p})$$

$$m = (m-C)$$

Now physical mass m_{phy} can be inferred by locating pole on the propagator

$$\frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}$$

The pole appears at

$$p^2 = m^2 \Rightarrow p^2 - m^2 = 0$$

or

$$(1-2A)E^2 - (1-2B)\mathbf{p}^2 - m^2 + 2mC = 0$$

which gives

$$E^2 - p^2 = m^2 + 2mC + 2AE^2 - 2B\mathbf{p}^2 = 0$$

$$= m^2 - \frac{\alpha}{4\pi^2}(2I.p + 2J_B.p + 2m^2 J_A)$$

$$= m^2_{phy}(p^2)$$

Therefore

$$m^2_{phy}(p^2) = m^2 + \frac{2}{3}\alpha\pi T^2 + \frac{\alpha}{2\pi^2} m^2 J_A + \frac{4\alpha}{\pi}\int_0^\infty \frac{l^2 dl}{E_l} n_F(E_l)$$

In a general FT framework, the last integral on RHS, gives

$$\int l^2 dl \frac{n_F(E_l)}{E_l} \approx \frac{m}{\beta} a(m\beta) - \frac{m^2}{2} b(m\beta) - \frac{1}{\beta^2} c(m\beta)$$

whereas

$$J_A = -8\pi b(m\beta)$$

with

$$a(m\beta) = \ln(1 + e^{-m\beta})$$

$$b(m\beta) = \sum_{n=1}^{\infty} (-1)^n \,\mathrm{Ei}(nm\beta)$$

and

$$c(m\beta) = \sum_{n=1}^{\infty} (-1)^n \frac{e^{-nm\beta}}{n^2}$$

Thus

$$m_{phy}^2(p^2) = m^2\left[1 - \frac{6\alpha}{\pi} b(m\beta)\right] + \frac{4\alpha}{\pi} mTb(m\beta) + \frac{2}{3}\alpha\pi T^2\left[1 - \frac{6}{\pi^2} c(m\beta)\right]$$

Radiative Corrections to Electron Propagator

- The shifted electron mass because of interaction with FT background $\delta m = m_{phy} - m$ becomes

$$\frac{\delta m}{m} = \frac{\alpha\pi T^2}{3m^2}\left[1 - \frac{6}{\pi^2}c(m\beta)\right] + \frac{2\alpha}{\pi}\frac{T}{m}a(m\beta) - \frac{3\alpha}{\pi}b(m\beta)$$

At low temperature $T << m$, giving mass shift:

$$\frac{\delta m}{m} \cong \frac{\alpha\pi T^2}{3m^2}$$

At very high temperature $T > m$:

$$\frac{\delta m}{m} \cong \frac{\alpha\pi T^2}{2m^2} - \frac{3\alpha}{\pi}S(N,\varepsilon) + \frac{2\alpha T}{\pi m}\ln 2$$

and

$$S(N,\varepsilon) = \sum_{n=1}^{\infty}(-1)^n \mathrm{Ei}(-n\varepsilon) \quad \text{with} \quad \varepsilon = m\beta << 1$$

taking $c(m\beta) \rightarrow -\frac{\pi^2}{12}$ and $a(m\beta) = \ln 2$

T^2 contribution becomes large enough giving $\frac{\delta m}{m} \cong \frac{\alpha\pi T^2}{2m^2}$

Scatterings at Finite Temperature

Low Temperature Rutherford Scattering

- Consider the case for T<<m due to background heat bath containing $e^{\pm}$ pairs

Lowest Order Correction

- Let four-momentum of incoming and scattered electron be

$$p_{1\mu} = (E_1, \mathbf{p}_1) \qquad \text{while} \qquad p_{2\mu} = (E_2, \mathbf{p}_2)$$

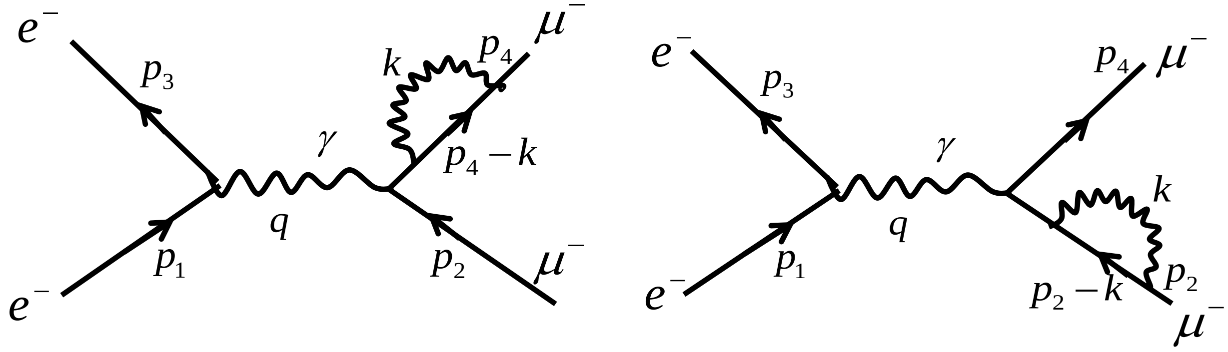
- The Coulomb field be

$$\varphi_\mu(|\mathbf{q}|) = u_\mu \frac{Ze}{|\mathbf{q}|^2} 2\pi\delta(\mathbf{q}_0) \quad \text{where} \quad \mathrm{q} = p_2 - p_1 \quad \text{and} \quad u^2 = 1$$

- The cross section for the lowest order into a solid angle dΩ is given by

$$d\sigma = 2\frac{Z^2\alpha^2}{|\mathbf{q}|^4} \sum_{spins} \left| \bar{u}(p_2) u_\mu \gamma^\mu u(p_1) \right|^2 \delta(E_2 - E_1) dE_2 d\Omega_2$$

Mott Scattering at Low Temperature



$$M = \int \{[\bar{u}(p_2)\frac{ie^2}{q^2}\gamma^\sigma u(p_1)][\bar{u}(p_3)\Sigma(p_3)\gamma_\sigma u(p_4)] + [\bar{u}(p_2)\frac{ie^2}{q^2}\gamma^\sigma u(p_1)][\bar{u}(p_3)\gamma_\sigma \Sigma(p_4)u(p_4)]\}$$

$$\times \delta^4(p_1 - p_2 - q)\delta^4(p_4 + q - p_3)d^4 q$$

Substituting for $\Sigma(p_3)$ and $\Sigma(p_4)$ in low temperature regime

$$M = \frac{ie^2}{q^2}\int \{[\bar{u}(p_2)\gamma^\sigma u(p_1)][\bar{u}(p_3)\gamma_\sigma \frac{\alpha}{4\pi^2}\{(\not{p}_4 - m_\mu)I_A + I(p_4)\}u(p_4)] + [\bar{u}(p_2)\gamma^\sigma u(p_1)]$$

$$[\bar{u}(p_3)\gamma_\sigma \frac{\alpha}{4\pi^2}\{(\not{p}_3 - m_\mu)I_A + I(p_3)\}u(p_4)]\} \times \delta^4(p_1 - p_2 - q)\delta^4(p_4 + q - p_3)d^4 q$$

or

$$M = \frac{e^2}{(p_1 - p_3)^2}\{[\bar{u}(p_2)\gamma^\sigma u(p_1)][\bar{u}(p_3)\gamma_\sigma \frac{\alpha}{4\pi^2}\{(\not{p}_4 - m_\mu)I_A + \not{I}(p_4)\}u(p_4)]$$

$$+[\bar{u}(p_2)\gamma^\sigma u(p_1)][\bar{u}(p_3)\gamma_\sigma \frac{\alpha}{4\pi^2}\{(\not{p}_3 - m_\mu)I_A + \not{I}(p_3)\}u(p_4)]\}$$

$\not{I}$ rearranged and written in terms of temperature as

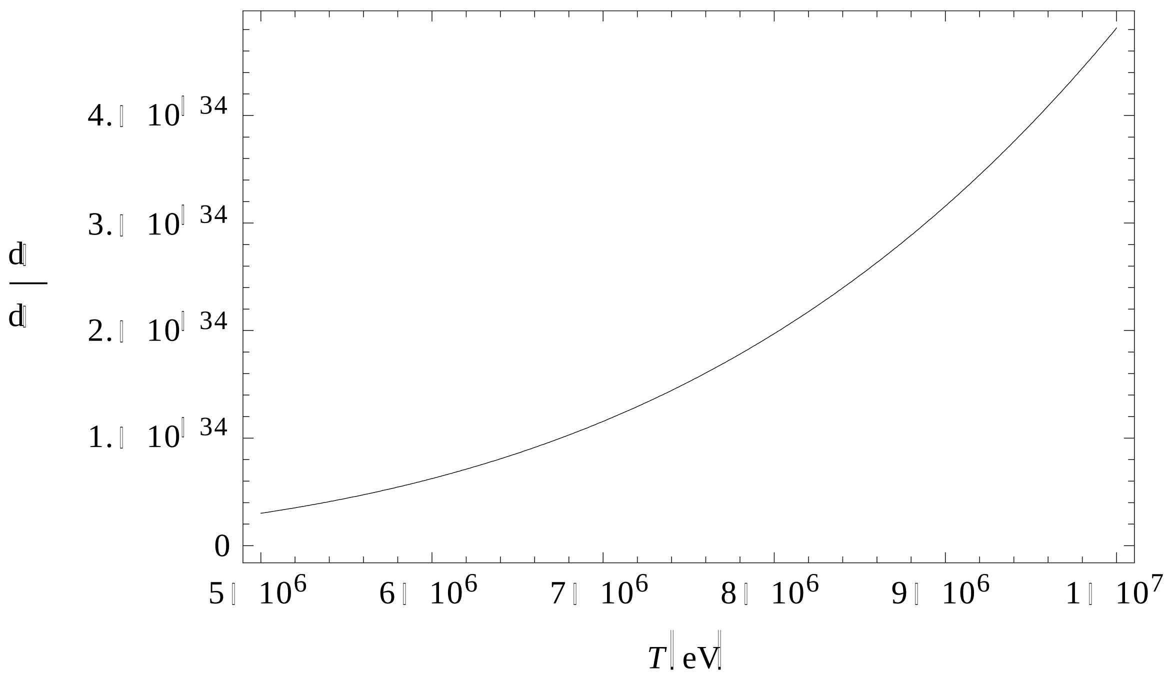
$$\frac{\alpha}{4\pi^2}\left(\frac{I_0(p)}{E} - \frac{I.\mathbf{p}}{|\mathbf{p}|^2}\right) = \frac{2\alpha\pi T^2}{3m_\mu{}^2}$$

Differential cross section becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\alpha^2}{2E^6}\left|\frac{\vec{p}}{\vec{k}}\right|\left(\frac{2\alpha\pi T^2}{3m_\mu{}^2}\right)^2 \left[4\left\{E^4 + (\mathbf{p.k})^2 + 2m^2 m_\mu{}^2\right\}\right]$$

Using the previous relations of **k** and **p** we get

$$\frac{d\sigma}{d\Omega}=\frac{8\alpha^4\pi^2T^4}{9m_\mu{}^4E^2}\left(1+\frac{1}{2}\frac{m^2}{E^2}-\frac{1}{2}\frac{m_\mu{}^2}{E^2}+..\right)\times\left[\left\{1+\left(1-\frac{m^2}{E^2}\right)\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta+2\frac{m_\mu{}^2m^2}{E^4}\right\}\right]$$



Mott Scattering at High Temperature

$$M = \int \{[\bar{u}(p_2)\frac{ie^2}{q^2}\gamma^\sigma u(p_1)][\bar{u}(p_3)\Sigma(p_3)\gamma_\sigma u(p_4)] + [\bar{u}(p_2)\frac{ie^2}{q^2}\gamma^\sigma u(p_1)][\bar{u}(p_3)\gamma_\sigma \Sigma(p_4) u(p_4)]\}$$

$$\times \delta^4(p_1 - p_2 - q)\delta^4(p_4 + q - p_3)d^4q$$

Which, at high temperature $T > m_\mu$

$$M = \frac{ie^2}{q^2}\int \{[\bar{u}(p_2)\gamma^\sigma u(p_1)]\delta^4(p_1 - p_2 - q)\delta^4(p_4 + q - p_3)d^4q$$

$$\times[\bar{u}(p_3)\gamma_\sigma \frac{\alpha}{4\pi^2}\{(\not{p}_4 - m_\mu)I_A + I(p_4) + (2m_\mu - \not{p}_4)J_A + \not{J}_B(p_4)\}u(p_4)]$$

$$+[\bar{u}(p_2)\gamma^\sigma u(p_1)]\times[\bar{u}(p_3)\frac{\alpha}{4\pi^2}\{(\not{p}_3 - m_\mu)I_A + \not{I}(p_3) + (2m_\mu - \not{p}_3)J_A + \not{J}_B(p_3)\}\gamma_\sigma u(p_4)]$$

FT contribution can be written as

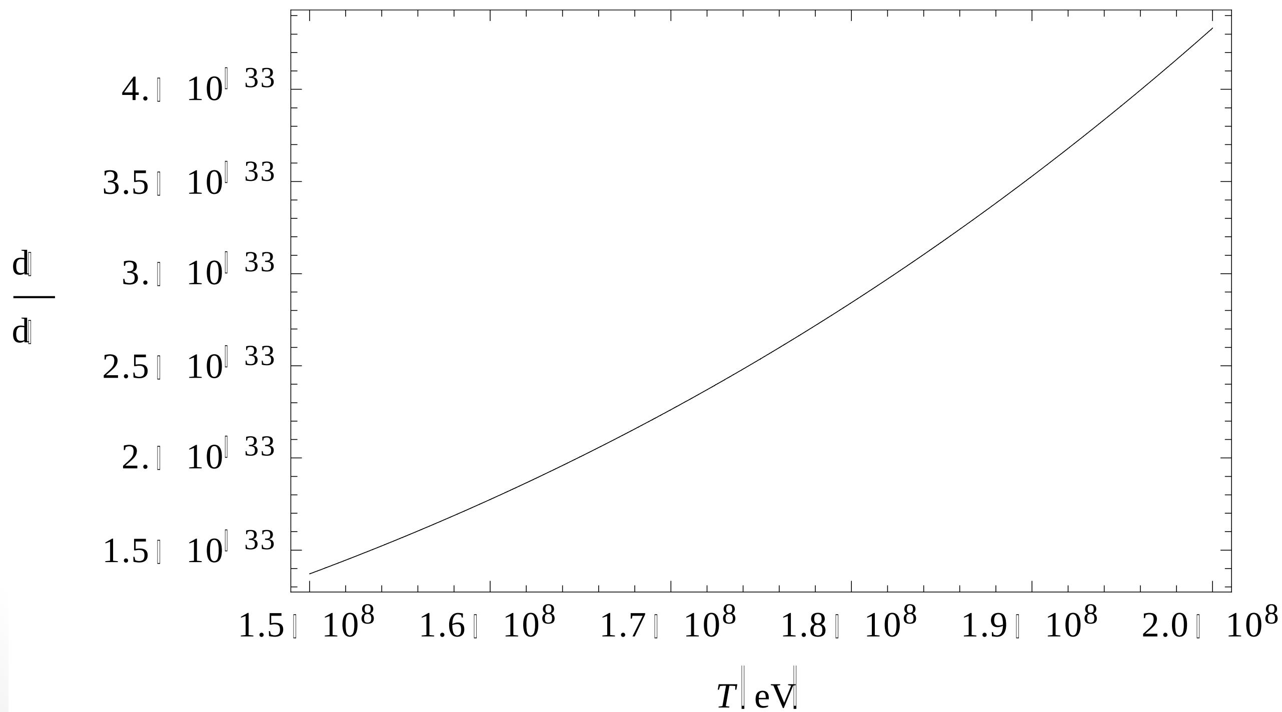
$$\frac{\alpha}{4\pi^2}\left(\frac{I_0(p)}{E} - \frac{I.\mathbf{p}}{|\mathbf{p}|^2} + J_A + \frac{J_0(p)}{E} - \frac{\mathbf{J.p}}{|\mathbf{p}|^2}\right) = \frac{\alpha\pi T^2}{2m_\mu^{\ 2}}$$

Differential cross section becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\alpha^2}{2E^6}\left|\frac{\vec{p}}{\vec{k}}\right|\left(\frac{\alpha\pi T^2}{2m_\mu{}^2}\right)^2 \left[4\left\{E^4 + (\mathbf{p.k})^2 + 2m^2 m_\mu{}^2\right\}\right]$$

Using the previous relations of **k** and **p** we get

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^4\pi^2 T^4}{2m_\mu{}^4 E^2}\left(1+\frac{1}{2}\frac{m^2}{E^2}-\frac{1}{2}\frac{m_\mu{}^2}{E^2}+..\right)\times\left[\left\{1+\left(1-\frac{m^2}{E^2}\right)\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta + 2\frac{m_\mu{}^2 m^2}{E^4}\right\}\right]$$



Summary

- Considered the differential cross sections for two types of QED scatterings, Rutherford and Mott Scattering separately.
- Reviewed cross sections in vacuum
- Calculated these scatterings by including FT statistical background by considering the presence of heat bath.
- Calculated one loop corrections to electron self energy in these scattering processes for FT using real time formalism.
- Then non relativistic Coulomb field was reviewed with the self energy correction at low temperature.
- The work was analytically done for FT corrections to Mott scattering, with lepton self energy corrections at
  - low temperature and
  - high temperature.

Results and Conclusions

- The differential cross section for the self energy corrections to Mott scattering at low temperatures, where only photons are considered at FT were calculated first calculated.
- The low temperature corrections to differential cross section are not completely negligible being about$10^{-34}m^{-2}$, as expected (cm energies 100GeV).
- We plotted the values of differential cross sections at low angle (15 degrees) with temperature background 150MeV to 200MeV and cm energy 10TeV (expected in future).
- We obtain high T contribution, around $4*10^{-33}m^{-2}$ to Mott scattering differential cross section to value in vacuum $10^{-30}m^{-2}$
- These results motivate us to look for similar effects
  - with vertex corrections to Mott scattering,
  - other elastic scatterings in QED and
  - also electroweak corrections.

THANK

YOU...