Techniques for multiloop calculations for the LHC

(a critical appraisal)

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TTP - KIT, Karlsruhe

International school of subnuclear physics - Erice, 17 June 2016

Collaboration with *T. Gehrmann, A. von Manteuffel, K. Melnikov, E. Remiddi*

Let's start from the beginning

Our understanding of high energy physics based mainly on
perturbative calculations in the Standard Model

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Comparison to experiments $\rightarrow$ [...], **Tevatron, LHC**, [...]

Tevatron's Legacy:
understanding fundamental laws
$\approx$ 10% accuracy!

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agreement with theory
NLO calculations!
$\propto$ one-loop calculations

LHC's goals:
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automation of one-loop corrections to high-multiplicities final states

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Many new results have become recently available it looks like we are **very close**

- Higgs production (up to N^3LO!!) ✔
- Higgs + jet production (Hj) ✔
- Vector boson + jet production (Zj, Wj) ✔
- Vector boson pair production (VV) ✔
- t-tbar production (ttb) ✔
- Higgs pair production (HH) – massive top loops! ✔

In reality we can handle only **massless QCD** corrections at this precision!

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From NLO → NNLO **jump in complexity** which we do not fully understand yet

$\mathcal{I}(p_1, p_2, q_1) =$ with $q_2 = p_1 + p_2 - q_1$

A (*possible*) representation in momentum space

$$\mathcal{I}(p_1, p_2, q_1) = \int \frac{\mathfrak{D}^d k \, \mathfrak{D}^d l}{(k^2 - m_1^2)\,(l^2 - m_2^2)\,(k-l)^2\,(k-p_1)^2\,(k-p_{12})^2\,(l-p_{12})^2\,(l-q_1)^2}$$

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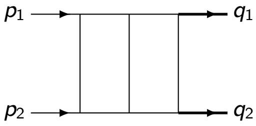

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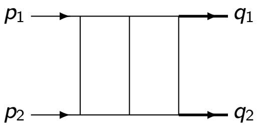

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How do we handle this complexity?

Integration-by-parts identities (IBPs)	**Differential Equations** (DEs)
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Reduce the number of integrals to compute	Expose their analytic structure

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This is all heavily based on Dimensional Regularization!
Dimensionally regularized Feynman integrals always converge!

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Integration By Parts Identities (IBPs) – *Gauss theorem* in d-dimensions

$$\int \prod_{j=1}^{l} \frac{d^d\, k_j}{(2\pi)^d} \left(\frac{\partial}{\partial k_j^{\mu}}\, v_{\mu}\, \frac{S_1^{\sigma_1} \dots S_s^{\sigma_s}}{D_1^{\alpha_1} \dots D_n^{\alpha_n}} \right) = 0\,, \qquad v^{\mu} = k_j^{\mu}, p_k^{\mu}$$

They generate *huge systems of linear equations* which relate similar integrals with *different powers* of numerators and denominators.

The problem becomes **algebraic**. Can we predict *how many integrals* will remain independent? Can we *solve* the system? Why in some cases integrals are completely **reduced** to *simpler integrals* (i.e. with less propagators) and in some other cases **master integrals** remain?

Very fascinating research area, mixing **algebra**, **algebraic geometry**, **topology**... [see for example arXiv:1507.06310 by Y. Zhang]

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Example IBPs: The 1loop tadpole

$$\mathcal{I}(n) = \int \frac{\mathfrak{D}^{\,d} k}{(k^2 + m^2)^n}$$

$$0 = \int \mathfrak{D}^{\,d} k \left(\frac{\partial}{\partial k_\mu} k_\mu \right) \frac{1}{(k^2 + m^2)^n} = (d - 2\,n)\mathcal{I}(n) + 2\,n\,m^2\,\mathcal{I}(n+1)$$

which gives for example:

$$(d-2)\mathcal{I}(1) + 2m^2\mathcal{I}(2) = 0 \quad \rightarrow \quad \mathcal{I}(2) = -\frac{(d-2)}{2m^2}\mathcal{I}(1)$$

$$(d-4)\mathcal{I}(2) + 4m^2\mathcal{I}(3) = 0 \quad \rightarrow \quad \mathcal{I}(3) = +\frac{(d-2)(d-4)}{8m^4}\mathcal{I}(1)$$

$$(d-6)\mathcal{I}(3) + 6m^2\mathcal{I}(4) = 0 \quad \rightarrow \quad \mathcal{I}(4) = -\frac{(d-2)(d-4)(d-6)}{48m^6}\mathcal{I}(1)$$

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From a more pragmatic point of view, we get symbolic system of (hundreds of) thousands of equations. Solution can produce **Gigabytes** of data or more!

Laporta Algorithm: *Classify equations and integrals* in order of complexity and solve them "*in the right order*". This allows to keep complexity under control, with a **large computer** we can go very far...

$$\Downarrow$$

Assuming we can do this step...

For $q\bar{q} \to ZZ$ production at 2 loops in QCD. We start with $\approx$ 8000 different integrals, we end up with $\approx$ **100 master integrals**.

It remains the issue of computing the master integrals!

Dimensionally regularised Feynman Integrals fulfil differential equations!
[Kotikov '90, Remiddi '97, Gehrmann-Remiddi '00,...]

This is a simple consequence of the IBPs.
We can differentiate any of the master integrals w.r.t. the external invariants x_k and then reduce it again to master integrals using IBPs.
Repeating the procedure for **every master** we end up with

$$\frac{\partial}{\partial x_k} m_i(d; x_k) = \sum_{j=1}^{N} c_{ij}(d; x_k)\, m_j(d; x_k)\,.$$

i.e. a *coupled system* of N linear differential equations with rational coefficients

The use of these techniques has made possible a huge number of two-loop $2 \to 2$ **massless** calculations!

Integration-by-parts identities (IBPs)

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Reduce the number of integrals to compute
In massless case, number of MIs is relatively small. Expressing physical results in terms of basis of MIs allows to reduce substantially the complexity of the problem!

Differential Equations (DEs)

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Expose their analytic structure In massless ($2 \to 2$) processes, a class of functions appears almost "ubiquitously"
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Differential equations and **multiple polylogarithms**

The MPLs have a very simple definition in terms of iterated integrals

$$G(0;x) = \ln x\,, \qquad G(a;x) = \ln\left(1 - \frac{x}{a}\right)$$

$$G(\vec{0}_n;x) = \frac{1}{n!}\ln^n x\,, \qquad G(a,\vec{n};x) = \int_0^x \frac{dt}{t-a}\,G(\vec{n};t)$$

Correct class of functions to express Feynman integrals as series in ϵ

$$\Downarrow$$

To see this, we go back to differential equation!

The basis of MIs is **not unique**!
It is often possible to find a basis such that deqs decouple as $\epsilon \to 0$

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When this is possible, one can *(usually)* do even better and put results in a very convenient form:

Canonical Basis [Kotikov '10; Henn '13]

$$\frac{\partial}{\partial x_k} \vec{\mathcal{I}}(\epsilon; x_k) = \epsilon\, A(x_k)\, \vec{\mathcal{I}}(\epsilon; x)\,, \quad \text{with } A(x_k) \text{ in d-log form}$$

Expanding right- and left-hand-side in series in ϵ, it is clear that order by order we get only multiple-polylogarithms (and "non-linear generlizations" thereof)

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Understanding of **analytic** and **algebraic** of MPLs revolutionized the field.
All $2 \to 2$ calculations completed, were possible using properties of MPLs.

1. Symbol and Co-Product formalism
   [Goncharov, Spradlin, Vergu, Volovich '10; Durh, Gangl, Rhodes '11; Duhr '12]

2. Of crucial importance,routines for their numerical evaluation on the *whole complex plane*!
   [Gehrmann, Remiddi '01; Vollinga, Weinzierl '04]

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Limitations of our two fundamental techniques

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2-loop massive corrections to $2 \to 2$ processes: H+jet, V+jet, VV, HH, ttbar Solution of IBPs becomes almost impossible.

Even if possible, remains impractical. Results are **GBs** or **TBs** or reduction identities!!!

Differential Equations (DEs)

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Deqs derived through IBPs. One is often left with $\approx$ 100MBs large differential equations!!!

Moreover, starting at two-loops, internal masses generate completely new mathematical structures: Elliptic integrals, integrals over elliptic integrals, ...

New mathematical structures found also in **string theory** (*one loop superstring amplitudes*). An entirely new branch of research is opening, putting together Mathematicians, Particle physicists and string theories, to understand the properties of these functions... **Elliptic Polylogarithms?** [Broadhurst; Remiddi, Laporta; Brown, Levine; Adams, Bogner, Weinzier; Remiddi, Tancredi ...]

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Conclusions

We are at a crossroad

1. Deeper and deeper understand of mathematical structure of loop amplitudes allowed huge step forward in **high energy phenomenology**

2. LHC physics would require study of massive corrections to different $2 \to 2$ processes.

3. **Analytical** approach to such calculations seems prohibitive

4. Where do we go from here? Need a **fresh look** at the problem and **fresh ideas** Unitarity at two-loops, new numerical methods which avoid complexity of symbolic expressions, new variables to enhance physics of the problem (twistor variables?)