Hamiltonian constraint formulation of classical field theories

Václav Zatloukal
[www.zatlovac.eu]

Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague

and

Max Planck Institute for the History of Science, Berlin

54th International School of Subnuclear Physics
Erice, 2016

Motivation

Consider a non-relativistic mechanical system with Hamiltonian $H_0(\mathbf{x}, \mathbf{p})$:

Motivation

Consider a non-relativistic mechanical system with Hamiltonian $H_0(\mathbf{x}, \mathbf{p})$:

Canonical equations of motion:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H_0}{\partial \mathbf{p}} \quad , \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H_0}{\partial \mathbf{x}} \tag{1}$$

Motivation

Consider a non-relativistic mechanical system with Hamiltonian $H_0(\mathbf{x}, \mathbf{p})$:

Canonical equations of motion:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H_0}{\partial \mathbf{p}} \quad , \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H_0}{\partial \mathbf{x}} \tag{1}$$

Hamilton-Jacobi equation: $S(\mathbf{x}, t)$

$$\frac{\partial S}{\partial t} + H_0(\mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}) = 0 \tag{2}$$

Motivation

Consider a non-relativistic mechanical system with Hamiltonian $H_0(\mathbf{x}, \mathbf{p})$:

Canonical equations of motion:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H_0}{\partial \mathbf{p}} \quad , \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H_0}{\partial \mathbf{x}} \tag{1}$$

Hamilton-Jacobi equation: $S(\mathbf{x}, t)$

$$\frac{\partial S}{\partial t} + H_0(\mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}) = 0 \tag{2}$$

Quantization & Schrödinger equation: $\mathbf{p} \to -i\hbar\,\partial/\partial\mathbf{x}$

$$\left[-i\hbar\frac{\partial}{\partial t} + H_0(\mathbf{x}, -i\hbar\frac{\partial}{\partial \mathbf{x}})\right]\psi(\mathbf{x}, t) = 0 \tag{3}$$

Motivation

Consider a non-relativistic mechanical system with Hamiltonian $H_0(\mathbf{x}, \mathbf{p})$:

Canonical equations of motion:

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H_0}{\partial \mathbf{p}} \quad , \quad \frac{d\mathbf{p}}{dt} = -\frac{\partial H_0}{\partial \mathbf{x}} \tag{1}$$

Hamilton-Jacobi equation: $S(\mathbf{x}, t)$

$$\frac{\partial S}{\partial t} + H_0(\mathbf{x}, \frac{\partial S}{\partial \mathbf{x}}) = 0 \tag{2}$$

Quantization & Schrödinger equation: $\mathbf{p} \to -i\hbar\,\partial/\partial\mathbf{x}$

$$\left[-i\hbar\frac{\partial}{\partial t} + H_0(\mathbf{x}, -i\hbar\frac{\partial}{\partial \mathbf{x}})\right]\psi(\mathbf{x}, t) = 0 \tag{3}$$

My goal today: Hamiltonian formulation of (classical) **field theory**

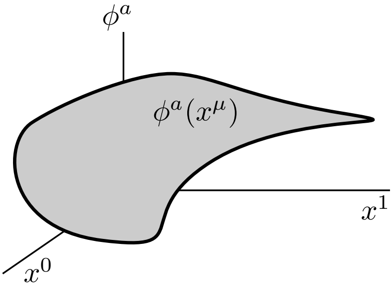
Outline

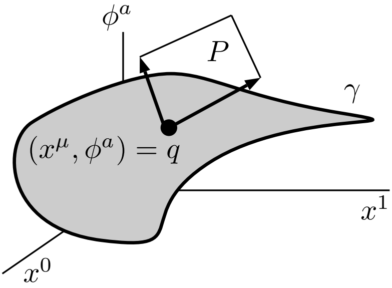
- Partial observables and generalized momentum
- Variational principle with Hamiltonian constraint
- Canonical equations of motion
- Local Hamilton-Jacobi theory
- Symmetries and Hamiltonian Noether theorem
- Examples:
  - Scalar field theory
  - String theory
- Discussion: Quantization

[V. Zatloukal, arXiv:1604.03974, arXiv:1602.00468, arXiv:1504.08344]

Partial observables and generalized momentum

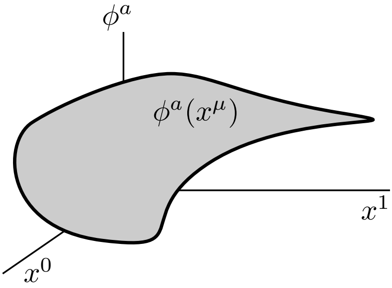
Functions $\phi^a(x^\mu) \to$ surfaces $\gamma = \{q = (x^\mu, \phi^a) \,|\, f(x,\phi) = 0\}$

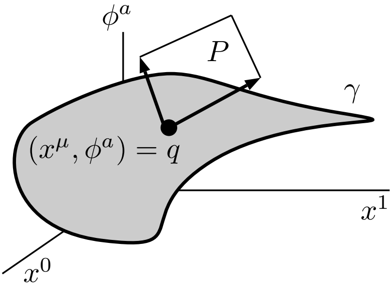




Partial observables and generalized momentum

Functions $\phi^a(x^\mu)$ → surfaces $\gamma = \{q = (x^\mu, \phi^a) \,|\, f(x, \phi) = 0\}$





$t, \mathbf{x}, \phi$ . . . *partial observables* [Rovelli]
$\mathcal{C} = \{q\}$ . . . *configuration space* – $N + D$-dimensional, Euclidean
$\gamma \subset \mathcal{C}$ . . . *motions* – D-dim. surfaces
($D = 1$: particle mechanics , $D > 1$: field theory)

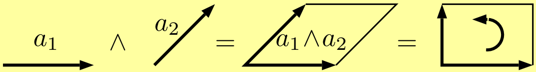
Geometric algebra formalism

Geometric algebra and calculus [Hestenes,Doran]
($\Leftrightarrow$ Clifford algebra, Dirac algebra of γ-matrices)

Geometric algebra formalism

Geometric algebra and calculus [Hestenes,Doran]
(⇔ Clifford algebra, Dirac algebra of γ-matrices)

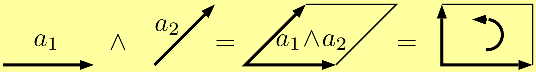
Geometric product: $ab = a \cdot b + a \wedge b$ (a, b vectors)



Geometric algebra formalism

Geometric algebra and calculus [Hestenes,Doran]
($\Leftrightarrow$ Clifford algebra, Dirac algebra of γ-matrices)

Geometric product: $ab = a \cdot b + a \wedge b$ (a, b vectors)

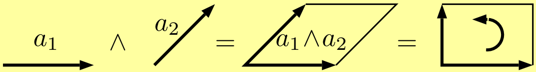


Vectors $a_1, \ldots, a_D$ $\rightarrow$ multivector $a_1 \wedge \ldots \wedge a_D$ of grade D

Geometric algebra formalism

Geometric algebra and calculus [Hestenes,Doran]
($\Leftrightarrow$ Clifford algebra, Dirac algebra of γ-matrices)

Geometric product: $ab = a \cdot b + a \wedge b$ (a, b vectors)



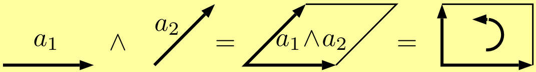
Vectors $a_1, \ldots, a_D$ $\rightarrow$ multivector $a_1 \wedge \ldots \wedge a_D$ of grade D

Differential forms: $\alpha(b_1, \ldots, b_D) := A \cdot (b_1 \wedge \ldots \wedge b_D)$ (A a D-vector)

Geometric algebra formalism

Geometric algebra and calculus [Hestenes,Doran]
($\Leftrightarrow$ Clifford algebra, Dirac algebra of γ-matrices)

Geometric product: $ab = a \cdot b + a \wedge b$ (a, b vectors)



Vectors $a_1, \ldots, a_D$ $\rightarrow$ multivector $a_1 \wedge \ldots \wedge a_D$ of grade D

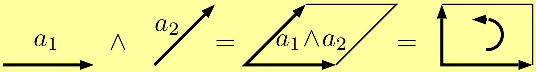
Differential forms: $\alpha(b_1, \ldots, b_D) := A \cdot (b_1 \wedge \ldots \wedge b_D)$ (A a D-vector)

Vector derivative: $\partial_q F(q) := \sum_{j=1}^{N+D} e_j (e_j \cdot \partial_q) F(q) = \underbrace{\partial_q \cdot F}_{\textit{divergence}} + \underbrace{\partial_q \wedge F}_{\textit{curl}}$

Geometric algebra formalism

Geometric algebra and calculus [Hestenes,Doran]
($\Leftrightarrow$ Clifford algebra, Dirac algebra of γ-matrices)

Geometric product: $ab = a \cdot b + a \wedge b$ (a, b vectors)



Vectors $a_1, \ldots, a_D$ $\rightarrow$ multivector $a_1 \wedge \ldots \wedge a_D$ of grade D

Differential forms: $\alpha(b_1, \ldots, b_D) := A \cdot (b_1 \wedge \ldots \wedge b_D)$ (A a D-vector)

Vector derivative: $\partial_q F(q) := \sum_{j=1}^{N+D} e_j (e_j \cdot \partial_q) F(q) = \underbrace{\partial_q \cdot F}_{divergence} + \underbrace{\partial_q \wedge F}_{curl}$

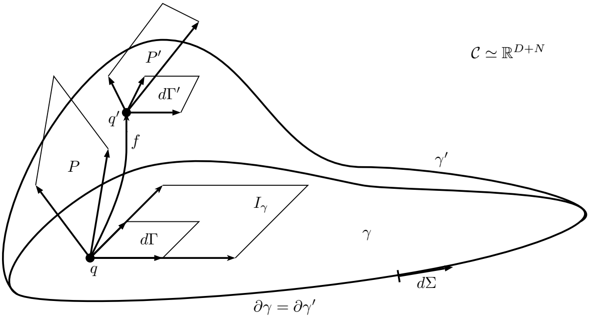
Multivector derivative: $\partial_P F(P) := \sum_{|J|=D} \widetilde{e}_J (e_J \cdot \partial_P) F(P)$

Variational principle with Hamiltonian constraint

$d\Gamma \ldots$ surface element of γ

$P \ldots$ momentum D-vect.

$\partial\gamma \ldots$ boundary of γ

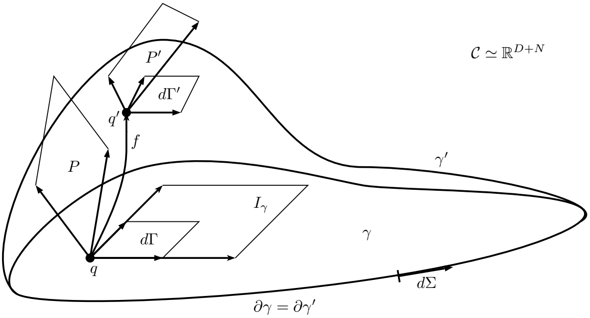


Variational principle with Hamiltonian constraint

$d\Gamma$... surface element of γ

P... momentum D-vect.

$\partial\gamma$... boundary of γ



Variational principle

Extremize the action $\mathcal{A}[\gamma, P] = \int_\gamma P(q) \cdot d\Gamma(q)$ *for fixed* $\partial\gamma$

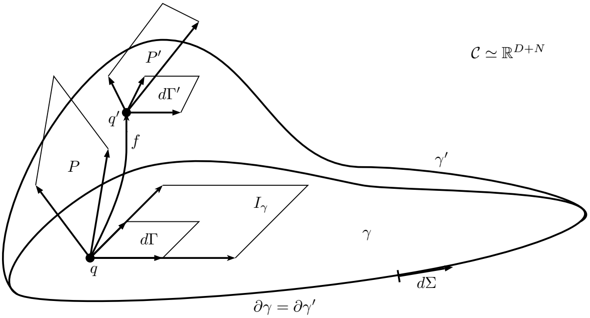
under the Hamiltonian constraint $H(q, P(q)) = 0$ $(\forall q \in \gamma)$. *[Rovelli]*

Variational principle with Hamiltonian constraint

$d\Gamma$... surface element of γ

P... momentum D-vect.

$\partial\gamma$... boundary of γ



Variational principle

Extremize the action $\mathcal{A}[\gamma, P] = \int_\gamma P(q) \cdot d\Gamma(q)$ *for fixed* $\partial\gamma$

under the Hamiltonian constraint $H(q, P(q)) = 0$ $(\forall q \in \gamma)$. *[Rovelli]*

Non-relativistic mechanics ... $H = p \cdot e_t + H_0(q, p_x)$
Scalar field theory ... $H = P \cdot I_x + \frac{1}{2} \sum_{a=1}^{N} \left(I_x \cdot (P \cdot e_a)\right)^2 + V(y)$
String theory ... $H = \frac{1}{2}(|P|^2 - \Lambda^2)$

Canonical equations of motion

($\lambda(q)$ . . . infinitesimal Lagrange multiplier)

$$\lambda\,\partial_P H(q,P) = d\Gamma \tag{4a}$$

$$(-1)^D \lambda\,\dot{\partial}_q H(\dot{q},P) = \begin{cases} d\Gamma \cdot \partial_q P & \text{for } D = 1 \\ (d\Gamma \cdot \partial_q) \cdot P & \text{for } D > 1, \end{cases} \tag{4b}$$

$$H(q,P) = 0. \tag{4c}$$

(4a) "Velocity–momentum" relation
(4b) "Force = Change of momentum"
(4c) Hamiltonian constraint

Local Hamilton-Jacobi theory

Assume $P(q) = \partial_q \wedge S(q)$ for a $D-1$-vector-valued function S.

Local Hamilton-Jacobi theory

Assume $P(q) = \partial_q \wedge S(q)$ for a $D-1$-vector-valued function S.

Hamiltonian constraint $\Rightarrow$

Local Hamilton-Jacobi equation

$$H(q, \partial_q \wedge S) = 0 \tag{5}$$

Local Hamilton-Jacobi theory

Assume $P(q) = \partial_q \wedge S(q)$ for a $D-1$-vector-valued function S.

Hamiltonian constraint $\Rightarrow$

Local Hamilton-Jacobi equation

$$H(q, \partial_q \wedge S) = 0 \tag{5}$$

IF (see Eq. (4a))

$$\lambda\, \partial_P H(q, \partial_q \wedge S) = d\Gamma \tag{6}$$

THEN Eq. (4b) holds automatically.

Local Hamilton-Jacobi theory

Assume $P(q) = \partial_q \wedge S(q)$ for a $D-1$-vector-valued function S.

Hamiltonian constraint $\Rightarrow$

Local Hamilton-Jacobi equation

$$H(q, \partial_q \wedge S) = 0 \tag{5}$$

IF (see Eq. (4a))

$$\lambda\, \partial_P H(q, \partial_q \wedge S) = d\Gamma \tag{6}$$

THEN Eq. (4b) holds automatically.

Continuous family of solutions $S(q; \alpha) \Rightarrow \partial_\alpha S$ conserved:

Local Hamilton-Jacobi theory

Assume $P(q) = \partial_q \wedge S(q)$ for a $D-1$-vector-valued function S.

Hamiltonian constraint $\Rightarrow$

Local Hamilton-Jacobi equation

$$H(q, \partial_q \wedge S) = 0 \tag{5}$$

IF (see Eq. (4a))

$$\lambda\, \partial_P H(q, \partial_q \wedge S) = d\Gamma \tag{6}$$

THEN Eq. (4b) holds automatically.

Continuous family of solutions $S(q;\alpha) \Rightarrow \partial_\alpha S$ conserved:

$D = 1$: $d\Gamma \cdot \partial_q(\partial_\alpha S) = 0 \quad \Rightarrow \quad \partial_\alpha S(q;\alpha) = const.$ along γ

Local Hamilton-Jacobi theory

Assume $P(q) = \partial_q \wedge S(q)$ for a $D-1$-vector-valued function S.

Hamiltonian constraint $\Rightarrow$

Local Hamilton-Jacobi equation

$$H(q, \partial_q \wedge S) = 0 \tag{5}$$

IF (see Eq. (4a))

$$\lambda\, \partial_P H(q, \partial_q \wedge S) = d\Gamma \tag{6}$$

THEN Eq. (4b) holds automatically.

Continuous family of solutions $S(q;\alpha) \Rightarrow \partial_\alpha S$ conserved:

$D = 1$: $d\Gamma \cdot \partial_q(\partial_\alpha S) = 0 \quad \Rightarrow \quad \partial_\alpha S(q;\alpha) = const.$ along γ

$D > 1$: $(d\Gamma \cdot \partial_q) \cdot (\partial_\alpha S) = 0$ ($\sim$ continuity equation)

Symmetries and Hamiltonian Noether theorem

Infinitesimal transformation $q \mapsto q + \varepsilon v(q)$ is a symmetry (i.e., maps classical motions to classical motions) IF

$$v \cdot \dot{\partial}_q H(\dot{q}, P) - \left(\dot{\partial}_q \wedge (\dot{v} \cdot P)\right) \cdot \partial_P H(q, P) = 0 \tag{7}$$

Symmetries and Hamiltonian Noether theorem

Infinitesimal transformation $q \mapsto q + \varepsilon v(q)$ is a symmetry (i.e., maps classical motions to classical motions) IF

$$v \cdot \dot{\partial}_q H(\dot{q}, P) - \left(\dot{\partial}_q \wedge (\dot{v} \cdot P)\right) \cdot \partial_P H(q, P) = 0 \tag{7}$$

$\oplus$ Canonical equations $\Rightarrow$

Conservation law

$$0 = \begin{cases} d\Gamma \cdot \partial_q (P \cdot v) & \text{for } D = 1 \\ (d\Gamma \cdot \partial_q) \cdot (P \cdot v) & \text{for } D > 1 \end{cases} \tag{8}$$

Integral form: $P(q_2) \cdot v(q_2) = P(q_1) \cdot v(q_1)$ *resp.* $\int_{\partial \gamma_{\mathrm{cl}}} d\Sigma \cdot (P \cdot v) = 0$

$P \cdot v$ ... conserved multivector of grade $D - 1$ ($\sim$ Noether current)

Example 1: Scalar field theory

Split $\mathcal{C} = \text{spacetime} \oplus \text{field space}$ ($q = x + y$), and consider

$$H_{SF}(q, P) = P \cdot I_x + \frac{1}{2} \sum_{a=1}^{N} \left(I_x \cdot (P \cdot e_a) \right)^2 + V(y) \qquad (9)$$

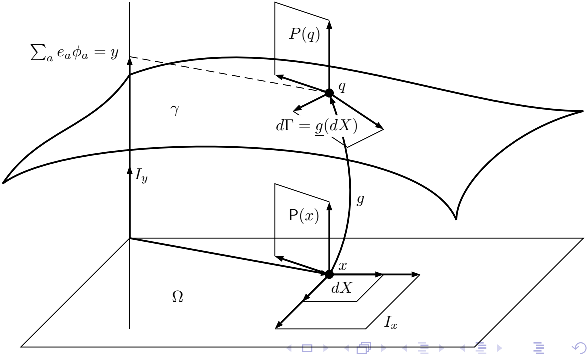
(I_x ... spacetime pseudoscalar, $\{e_a\}_{a=1}^{N}$ ... basis of the field space)

Functions on spacetime:

$\gamma = \{x + y(x) \,|\, x \in \Omega\}$

$\mathsf{P}(x) \equiv P(x + y(x))$

$\mathsf{v}(x) \equiv v(x + y(x))$



Example 1: Scalar field theory

Canonical equations: (cf. De Donder-Weyl equations)

$$\partial_x y = I_x \sum_{a=1}^{N} e_a \wedge (e_a \cdot \mathrm{P}) \quad , \quad (e_a I_x \partial_x) \cdot \mathrm{P} = (-1)^D e_a \cdot \partial_y V(y) \tag{10}$$

Example 1: Scalar field theory

Canonical equations: (cf. De Donder-Weyl equations)

$$\partial_x y = I_x \sum_{a=1}^{N} e_a \wedge (e_a \cdot \mathrm{P}) \quad , \quad (e_a I_x \partial_x) \cdot \mathrm{P} = (-1)^D e_a \cdot \partial_y V(y) \tag{10}$$

Hamilton-Jacobi equation: (cf. Weyl's HJ eq.)

$$I_x \cdot (\partial_x \wedge S) + \frac{1}{2} \sum_{a=1}^{N} \big(I_x \cdot ((\partial_q \wedge S) \cdot e_a)\big)^2 + V(y) = 0 \tag{11}$$

Example 1: Scalar field theory

Canonical equations: (cf. De Donder-Weyl equations)

$$\partial_x y = I_x \sum_{a=1}^{N} e_a \wedge (e_a \cdot \mathrm{P}) \quad , \quad (e_a I_x \partial_x) \cdot \mathrm{P} = (-1)^D e_a \cdot \partial_y V(y) \qquad (10)$$

Hamilton-Jacobi equation: (cf. Weyl's HJ eq.)

$$I_x \cdot (\partial_x \wedge S) + \frac{1}{2} \sum_{a=1}^{N} \left(I_x \cdot ((\partial_q \wedge S) \cdot e_a)\right)^2 + V(y) = 0 \qquad (11)$$

Conservation law $(d\Gamma \cdot \partial_q) \cdot (P \cdot v) = 0 \quad \Rightarrow$ Continuity equation:

$$\partial_x \cdot j(x) = 0 \qquad (12)$$

with the Noether current $j(x) \equiv -I_x \cdot \left[\mathrm{P} \cdot \mathrm{v} + \dot{\partial}_x \wedge \left(\dot{y} \cdot (\mathrm{P} \cdot \mathrm{v})\right)\right]$

Example 1: Scalar field theory

Canonical equations: (cf. De Donder-Weyl equations)

$$\partial_x y = I_x \sum_{a=1}^{N} e_a \wedge (e_a \cdot \mathrm{P}) \quad , \quad (e_a I_x \partial_x) \cdot \mathrm{P} = (-1)^D e_a \cdot \partial_y V(y) \tag{10}$$

Hamilton-Jacobi equation: (cf. Weyl's HJ eq.)

$$I_x \cdot (\partial_x \wedge S) + \frac{1}{2} \sum_{a=1}^{N} \left(I_x \cdot ((\partial_q \wedge S) \cdot e_a) \right)^2 + V(y) = 0 \tag{11}$$

Conservation law $(d\Gamma \cdot \partial_q) \cdot (P \cdot v) = 0 \quad \Rightarrow$ Continuity equation:

$$\partial_x \cdot j(x) = 0 \tag{12}$$

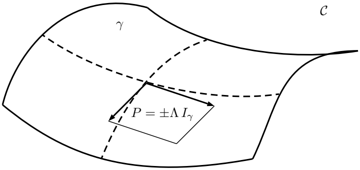
with the Noether current $j(x) \equiv -I_x \cdot \left[\mathrm{P} \cdot \mathrm{v} + \dot{\partial}_x \wedge \left(\dot{y} \cdot (\mathrm{P} \cdot \mathrm{v}) \right) \right]$

(E.g., spacetime translations $v(q) = v_x \;\rightarrow\; j(x; v_x)$ ... energy-momentum tensor)

Example 2: String theory

$\mathcal{C}$ ... target space
γ ... world-sheet

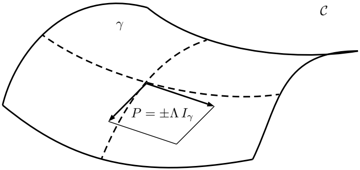
$$H_{Str}(P) = \frac{1}{2}(|P|^2 - \Lambda^2)$$



Example 2: String theory

$\mathcal{C}$ ... target space
γ ... world-sheet

$$H_{Str}(P) = \frac{1}{2}(|P|^2 - \Lambda^2)$$



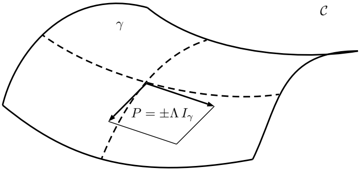
Canonical equations imply:

$$d\Gamma = \lambda \widetilde{P} \quad , \quad |d\Gamma| = |\lambda|\Lambda$$

Example 2: String theory

$\mathcal{C}$ ... target space
γ ... world-sheet

$$H_{Str}(P) = \frac{1}{2}(|P|^2 - \Lambda^2)$$



Canonical equations imply:

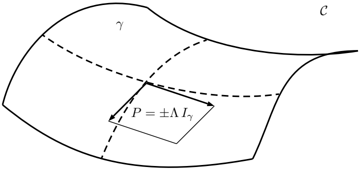
$$d\Gamma = \lambda \widetilde{P} \quad , \quad |d\Gamma| = |\lambda|\Lambda$$

Action: $\mathcal{A}_{Str} = \int_\gamma P \cdot d\Gamma = \pm\Lambda \int_\gamma |d\Gamma|$ (cf. Nambu-Goto)

Example 2: String theory

$\mathcal{C}$ . . . target space
γ . . . world-sheet

$$H_{Str}(P) = \frac{1}{2}(|P|^2 - \Lambda^2)$$



Canonical equations imply:

$$d\Gamma = \lambda \widetilde{P} \quad , \quad |d\Gamma| = |\lambda|\Lambda$$

Action: $\mathcal{A}_{Str} = \int_\gamma P \cdot d\Gamma = \pm\Lambda \int_\gamma |d\Gamma|$ (cf. Nambu-Goto)

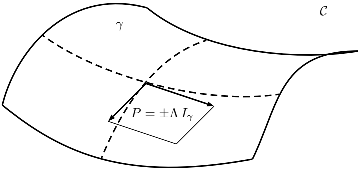
Eq. of motion: $(I_\gamma \cdot \partial_q) \cdot I_\gamma = 0$, $I_\gamma \equiv \frac{d\Gamma}{|d\Gamma|}$ . . . unit pseudoscalar of γ

Example 2: String theory

$\mathcal{C}$ ... target space

γ ... world-sheet

$$H_{Str}(P) = \frac{1}{2}(|P|^2 - \Lambda^2)$$



Canonical equations imply:

$$d\Gamma = \lambda\widetilde{P} \quad , \quad |d\Gamma| = |\lambda|\Lambda$$

Action: $\mathcal{A}_{Str} = \int_\gamma P \cdot d\Gamma = \pm\Lambda \int_\gamma |d\Gamma|$ (cf. Nambu-Goto)

Eq. of motion: $(I_\gamma \cdot \partial_q) \cdot I_\gamma = 0$, $I_\gamma \equiv \frac{d\Gamma}{|d\Gamma|}$ ... unit pseudoscalar of γ

Hamilton-Jacobi eq.: $|\partial_q \wedge S| = \Lambda$

Summary of results

- We have seen how field theory can be formulated using Hamiltonian constraint between partial observables and generalized momentum:
  $\mathcal{A} = \int_\gamma P \cdot d\Gamma \quad , \quad H(q, P) = 0$
- Canonical equations of motion:
  $\lambda\, \partial_P H(q, P) = d\Gamma \quad , \quad (-1)^D \lambda\, \dot{\partial}_q H(\dot{q}, P) = (d\Gamma \cdot \partial_q) \cdot P$
- Local Hamilton-Jacobi equation:
  $H(q, \partial_q \wedge S) = 0$
- Field-theoretic Hamiltonian Noether theorem:
  $(d\Gamma \cdot \partial_q) \cdot (P \cdot v) = 0$
- Two examples provided: Scalar field theory, String theory

Discussion: Quantization

Path integral:

$$\psi[\partial\gamma] = \int_{\partial\gamma \text{ fixed}} \mathcal{D}\gamma \mathcal{D}P \, e^{\frac{i}{\hbar}\int_\gamma P\cdot d\Gamma} \delta[H(q,P)] \tag{13}$$

Discussion: Quantization

Path integral:

$$\psi[\partial\gamma] = \int_{\partial\gamma \text{ fixed}} \mathcal{D}\gamma\mathcal{D}P\, e^{\frac{i}{\hbar}\int_\gamma P\cdot d\Gamma}\delta[H(q,P)] \tag{13}$$

"Canonical" approach:

classical momentum P $\rightarrow$

mechanics: $\hat{p} = -i\hbar\partial_q$

field theory: $\hat{P} = (?)$

Discussion: Quantization

Path integral:

$$\psi[\partial\gamma] = \int_{\partial\gamma \text{ fixed}} \mathcal{D}\gamma \mathcal{D}P \, e^{\frac{i}{\hbar}\int_\gamma P\cdot d\Gamma} \delta[H(q,P)] \tag{13}$$

"Canonical" approach:

classical momentum P $\rightarrow$

mechanics: $\hat{p} = -i\hbar\partial_q$

field theory: $\hat{P} = (?)$

Hamilton-Jacobi eq. $\rightarrow$
$H(q, \partial_q \wedge S(q)) = 0$

mechanics:
$H(q, -i\hbar\partial_q)\psi(q) = 0$
(Schrödinger, Klein-Gordon, ...)

field theory: $(?) = 0$
(Hints in [Kanatchikov])

Thank you for your attention.

References:

V. Zatloukal, arXiv:1604.03974, arXiv:1602.00468, arXiv:1504.08344

C. Rovelli, Chapter 3 in *Quantum Gravity* (2004)

D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus* (1987)

C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (2007)

I. V. Kanatchikov, arXiv:1312.4518