Imaginary part of Gaussian effective action of scalar field in de Sitter space

Akhmedov E.T., Bazarov K.V.

Moscow Institute of Physics and Technology
Institute for Theoretical and Experimental Physics


June 2019
Plan:

1) Introduction and literature
2) Motivation to calculation - particle creation
3) Time-ordered propagator
4) Another signals of instability
It’s stated in 0909.3722 , that there is a controversy in the literature on the issue of the presence of the Schwinger type imaginary contributions in the one–loop scalar effective actions in de Sitte space and on the physical implications of the corresponding pair production.
To calculate the imaginary contribution to the effective action one has to use the in–out Feynman propagator. Usually one uses the so called Bunch–Davies Feynman propagator, which is constructed for the scalar, $\varphi$, field theory as follows

$$G_{BD}(x,y) = \langle BD \left| T\varphi(x)\varphi(y) \right| BD \rangle,$$

where $\left| BD \right\rangle$ is the Bunch–Davies vacuum. In this case one does not find any imaginary Schwinger type contribution to the one loop effective action.
But there is an opinion that $|\text{BD}\rangle$ isn’t correct ground state, because, as it’s stated in 0709.2899, the asymptotic condition which uniquely fixes the propagator is

$$G(x, y) \sim e^{-imL(x, y)},$$

(1)

where $L(x, y) \to \infty$ being the geodesic time-like length. So we have to construct suitable states, which leads to correct behaviour of propagator (1).
Motivation

We denote the mass of massive real scalar field $\varphi$ as $m$. Because of the relation:

$$\langle \text{Out} | \text{In} \rangle = e^{i \int \mathcal{L}_{\text{eff}} dx}, \quad \text{and} \quad \mathcal{L}_{\text{eff}} = \int_{\infty}^{m^2} dm^2 \ G_F(x, x),$$

if $\mathcal{L}_{\text{eff}}$ is real this transition amplitude will be some phase and the probability of the transition from the In- to the Out- state is equal to one. But if the effective Lagrangian has an imaginary part the probability of such a transition is not equal to one:

$$\left| \langle \text{Out} | \text{In} \rangle \right|^2 \neq 1,$$

which usually signals a particle creation!
Space-time, metric and equation of motion

Consider D-dimensional global de Sitter space \((R = 1)\) with the following metric:
\[
ds^2 = -dt^2 + \cosh^2(t) d\Omega^2.
\]

Klein-Gordon equation:
\[
\left(\partial_t^2 + (D - 1) \tanh(t) \partial_t + j(j + D - 2) \cosh^{-2}(t) + m^2\right) \varphi_j(t) = 0.
\]

This equation has two linear independent solutions:
\[
\varphi_j(t) = \alpha_1 \text{ch}(t)^{-\frac{D-1}{2}} P_{j+\frac{D-3}{2}}^{-i\mu}(\tanh t) + \alpha_2 \frac{2}{\pi} \text{ch}(t)^{-\frac{D-1}{2}} Q_{j+\frac{D-3}{2}}^{-i\mu}(\tanh t).
\]

Spherical harmonics expansion is performed
\[
\varphi = \sum_{j,m} \varphi_j(t) Y_{jm}(\Omega), \quad \mu^2 = m^2 - \frac{(D - 1)^2}{4}.
\]
Here and below $\vec{x}$ is a vector of angular coordinates on $(D - 1)$-dimensional sphere. Consider the field operator ($\tilde{t} \equiv \tanh t$):

$$\hat{\varphi}(t, \vec{x}) = \sum_{j,m} \text{ch}(t)^{-\frac{D-1}{2}} \left[ \left( \alpha_1 P^\text{−i}_\nu (\tilde{t}) + \alpha_2 \frac{2}{\pi} Q^\text{−i}_\nu (\tilde{t}) \right) Y_{jm}(\vec{x}) \hat{a}_{j,m}^\dagger + h.c. \right] .$$

Annihilation and creation operators:

$$[\hat{a}_{j,m}, \hat{a}_{j',m'}^\dagger] = \delta_{j,j'} \delta_{m,m'} .$$

Canonical commutation relations:

$$[\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{y})] = i \frac{\delta(\vec{x} - \vec{y})}{\sqrt{g}} \rightarrow$$

$$\left( |\alpha_1|^2 + |\alpha_2|^2 \right) \frac{2i \sinh(\mu \pi)}{\pi} - \left( \alpha^*_1 \alpha_2 - \alpha^*_2 \alpha_1 \right) \frac{2 \cosh(\mu \pi)}{\pi} = i$$

This is the condition, which should be obeyed by $\alpha_{1,2}$ coefficients to have the canonical commutation relations.
Behavior of modes at plus and minus infinity

One can find asymptotic expansion for modes. For example:

\[ P_{\nu}^{-i\mu}(\tanh t) \approx C_+ e^{i\mu t} + C_- e^{-i\mu t}, \quad \text{as} \quad t \to -\infty. \]

So, modes behave like waves at \( t \to \pm \infty \). We are interested in single wave behavior. Hence one wave at plus infinity (Out-modes) corresponds to:

\[ \alpha_1 = \sqrt{\frac{\pi}{2 \sinh(\mu \pi)}}, \quad \text{and} \quad \alpha_2 = 0. \]

At the same time the one wave at minus infinity (In-modes) corresponds to:

\[ \alpha_1 = \sqrt{\frac{\pi}{2 \sinh(\mu \pi)}}, \quad \alpha_2 = 0, \quad \text{in odd dimensions}, \]

and \[ \alpha_2 = \sqrt{\frac{\pi}{2 \sinh(\mu \pi)}}, \quad \alpha_1 = 0, \quad \text{in even dimensions}. \]
Out- mode even

In- mode even
In- and Out- mode odd
Vacuum states

Vacuum state is defined as:

\[ a_{j,m} |\alpha\rangle = 0. \]

It means that:

different \( \alpha_1, \alpha_2 \rightarrow \) different mode expansion \( \rightarrow \)
different creation and annihilation operators \( \rightarrow \) different ground states.

We consider two states:

\[ |\text{In}\rangle \quad \text{single wave at past infinity} \]

\[ |\text{Out}\rangle \quad \text{single wave at future infinity} \]
Feynman In-Out propagator in even dimensions

\[ G_{\text{In-Out}}(t_1, \vec{x} | t_2, \vec{y})^{\text{even}} = \frac{\langle \text{Out} | T \hat{\phi}(\vec{y}, t_2) \hat{\phi}(\vec{x}, t_1) | \text{In} \rangle}{\langle \text{Out} | \text{In} \rangle} = \]

\[ = - \frac{i(-1)^{\frac{D-2}{2}}}{2(2\pi)^{\frac{D}{2}} \text{ch} \mu \pi} \left[ \left( Z_+^2 - 1 \right)^{\frac{D-2}{4}} Q^{\frac{D-2}{2}}_{-i\mu - \frac{1}{2}} (-Z_+) + \left( Z_-^2 - 1 \right)^{\frac{D-2}{4}} Q^{\frac{D-2}{2}}_{-i\mu - \frac{1}{2}} (-Z_-) \right] \]

\[ \text{Im} \ G_{\text{In-Out}}^{\text{even}}(Z = 1) = - \frac{(-1)^{\frac{D-2}{2}} |\Gamma(\frac{D-1}{2} + i\mu)|^2}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2}) \cosh \pi \mu}. \]

\[ Z_\pm \equiv Z \pm i\epsilon = - \frac{\tanh t_1 \tanh t_2 + \vec{x} \vec{y}}{\sqrt{1 - (\tanh t_1)^2} \sqrt{1 - (\tanh t_2)^2}} \pm i\epsilon. \]
Another interpretation

We saw that answer has significant dependence on the number of dimensions. And one can see another argument of validity of this fact. One can convert effective action into the quantum mechanical path integral:

\[ iS_{\text{eff}} = \log \left( \int d[\varphi] e^{i \int d^d x \mathcal{L}} \right) = \int_0^\infty \frac{dT}{T} \int_{x(0)=x(T)} d[x] e^{i \int_0^T dt \left( \frac{\dot{x}^2}{4} + m^2 \right)} = \]

\[ = \int_0^\infty \frac{dT}{T} e^{iS_{\text{extremal}}} \sqrt{\frac{(2\pi i)^d}{\det (\Delta_1)}} , \]

Usually one calculates such an integral via the Wick rotating from de Sitter to Euclidean sphere, one obtain that geodesic on sphere is equator.
Another interpretation

On the D-sphere exist (D-1) direction to shrink the geodesic, that corresponds to the fact that there are (D-1) negative eigenvalue. So:

\[-S_{\text{eff}}^E \sim \sqrt{\det (\Delta_1)} \sim (-1)^{\frac{d-1}{2}}.\]

Consequently for even dimension \(\text{Im} \left[ (S_{\text{eff}}) \neq 0 \right]\) and vanish for odd.
Thank you for your attention!