The limit shape phenomena in representation theory of quantum groups

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Main result

We obtain the formula for multiplicities of irreducible component \([V_l]\) in tensor product decomposition of \(U_q(sl_2)\)-moduli \([V_1]^N\), where \(q = e^{2\pi i r}\) is a root of unity:

\[
m_l^{(r)}(N) = \frac{N!(l + 1)}{\left(\frac{N-l}{2}\right)!\left(\frac{N+l}{2} + 1\right)!} + \sum_{k=1}^{[\frac{N}{2r}]} \frac{N!(l - 2kr + 1)}{\left(\frac{N-l}{2} + kr\right)!\left(\frac{N+l}{2} - kr + 1\right)!} + \sum_{k=1}^{[\frac{N}{2r}]} \frac{N!(l + 2kr + 1)}{\left(\frac{N-l}{2} - kr\right)!\left(\frac{N+l}{2} + kr + 1\right)!}
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+ \sum_{k=1}^{[\frac{N}{2r}]} \frac{N!(l + 2kr + 1)}{(\frac{N-l}{2} - kr)!(\frac{N+l}{2} + kr + 1)!}
\]

We derive the limit of the formula above, with respect to \(r, N, l \to \infty, \lambda = \frac{l}{r}\) and \(\tau = \frac{N}{r}\) remain fixed:

\[
m_l^{(r)}(N) \simeq \frac{2\lambda}{\tau + \lambda} \sqrt{\frac{2\tau}{\pi r(\tau - \lambda)(\tau + \lambda)}} e^{r\left(ln\tau - (\frac{\tau-\lambda}{2})ln(\frac{\tau-\lambda}{2}) - (\frac{\tau+\lambda}{2})ln(\frac{\tau+\lambda}{2})\right)} \left(1 + o\left(\frac{1}{r}\right)\right)
\]
Motivation

Possible applications:

Representation theory of $U_q(sl_2)$ plays important role in solving quantum spin chains (Statistics of solutions of the Bethe equations, thermodynamical limit)

Multiplicities of irreducible components are of great importance in topological quantum field theory (Volume conjecture: $\lim_{r \to 1} \ln |V_r(K; q = e^{2i\pi r})| = Vol(S^3 = K)$)

Asymptotic representation theory can be applied to different fields of theoretical physics, especially where integrability emerges (Precision test in duality between type IIB string theory in AdS$_5$S$_5$ and $N=4$ SYM)
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  (Precision test in duality between type $IIB$ string theory in $AdS_5 \times S^5$ and $\mathcal{N}=4$ SYM)
Introduction to the problem

Definition

$U_q(sl_2(\mathbb{C}))$ is a $q$-deformation of the universal enveloping algebra of the Lie algebra $sl_2(\mathbb{C})$, where $q = e^{\frac{2\pi i}{r}}$. Its $(l+1)$-dimensional vector representation is denoted by $V_l$ and corresponding isomorphism class by $[V_l]$.

Truncated tensor product rule

\begin{align*}
[V_1] \cdot [V_0] &= [V_1] \quad (1) \\
[V_1] \cdot [V_l] &= [V_{l+1}] \oplus [V_{l-1}], \quad l = 1, \ldots, r - 2 \quad (2)
\end{align*}

Tensor product decomposition

\begin{align*}
[V_1]^N &= \bigoplus_{l=0}^{r-2} m^{(r)}_l(N) \cdot [V_l] \quad (3)
\end{align*}
Instead of using algebraic framework, we consider random walk on the lattice, where $N$ is time, $l$ is a coordinate of a particle, $m_{i}^{(r)}(N)$ is a number of paths, which descend from $(0, 0)$ to $(l, N)$ in the region $l \in (0, r - 2)$. The dynamics of the particle corresponds to the truncated tensor product rule.
Example: $r = 9$
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$$[V_1]^7 = 14[V_1] \oplus 14[V_3] \oplus 6[V_5] \oplus [V_7]$$
The number of such paths was obtained by means of the inversion principle, involution method and corresponding framework.\textsuperscript{1}

\[ m_l^{(r)}(N) = \frac{N!(l+1)}{\left(\frac{N-l}{2}\right)!\left(\frac{N+l}{2}+1\right)!} + \sum_{k=1}^{\left[\frac{N}{2r}+\frac{1}{2}\right]} \frac{N!(l-2kr+1)}{\left(\frac{N-l}{2}+kr\right)!\left(\frac{N+l}{2}-kr+1\right)!} \]

\[ + \sum_{k=1}^{\left[\frac{N}{2r}\right]} \frac{N!(l+2kr+1)}{\left(\frac{N-l}{2}-kr\right)!\left(\frac{N+l}{2}+kr+1\right)!} \]

The limit shape

The limit of the discrete formula, where \( r, N, l \to \infty \) and \( \lambda = \frac{l}{r} \), \( \tau = \frac{N}{r} \) remain fixed, was derived by means of Stirling formula

\[
\begin{align*}
    m_r^r(N) &\simeq \left( \frac{2\lambda}{\tau + \lambda} \sqrt{\frac{2\tau}{\pi r (\tau - \lambda) (\tau + \lambda)}} \right) e^{r \left( \tau \ln \tau - \left( \frac{\tau - \lambda}{2} \right) \ln \left( \frac{\tau - \lambda}{2} \right) - \left( \frac{\tau + \lambda}{2} \right) \ln \left( \frac{\tau + \lambda}{2} \right) \right)} \\
    &+ \sum_{k=1}^{[\frac{\tau+1}{2}]} \frac{2(\lambda - 2k)}{\tau + \lambda - 2k} \sqrt{\frac{2\tau}{\pi r (\tau - \lambda + 2k)(\tau + \lambda - 2k)}} e^{r \left( \tau \ln \tau - \left( \frac{\tau - \lambda}{2} + k \right) \ln \left( \frac{\tau - \lambda}{2} + k \right) - \left( \frac{\tau + \lambda}{2} - k \right) \ln \left( \frac{\tau + \lambda}{2} - k \right) \right)} \\
    &+ \sum_{k=1}^{[\frac{\tau}{2}]} \frac{2(\lambda + 2k)}{\tau + \lambda + 2k} \sqrt{\frac{2\tau}{\pi r (\tau - \lambda - 2k)(\tau + \lambda + 2k)}} e^{r \left( \tau \ln \tau - \left( \frac{\tau - \lambda}{2} - k \right) \ln \left( \frac{\tau - \lambda}{2} - k \right) - \left( \frac{\tau + \lambda}{2} + k \right) \ln \left( \frac{\tau + \lambda}{2} + k \right) \right)} \\
    &\times \left( 1 + o \left( \frac{1}{r} \right) \right)
\end{align*}
\]
We show, that the second and the third terms can be neglected, as functions inside the exponents provide smaller growth.

Fourth triangle: yellow colour - the first term, blue - second, green - third, red - fourth. \( \tau \in [3, 5], \lambda \in [0, 1] \)
Final result

Hence, we arrive at

$$m_i^{(r)}(N) \simeq \frac{2\lambda}{\tau + \lambda} \sqrt{\frac{2\tau}{\pi r(\tau - \lambda)(\tau + \lambda)}} e^{r\left(\tau \ln \tau - \left(\frac{\tau - \lambda}{2}\right) \ln \left(\frac{\tau - \lambda}{2}\right) - \left(\frac{\tau + \lambda}{2}\right) \ln \left(\frac{\tau + \lambda}{2}\right)\right)}$$
Concluding remarks

- We obtained the formula for multiplicities of irreducible components in tensor product decomposition of $U_q(sl_2)$-module, where $q$ is a root of unity.
- We derived the limit of such formula, where $r \rightarrow \infty$ and $\lambda = \frac{l}{r}$, $\tau = \frac{N}{r}$ remain fixed, which gives us its asymptotical behavior. Surprisingly, the walk doesn’t feel the presence of the restrictions.
Thank you for the attention!